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Pattern Formation, Spatial Externalities and Regulation in Coupled Economic-Ecological Systems

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Abstract

We develop a novel theoretical framework for studying ecosystems in which interacting state variables which are affected by management decisions diffuse in space. We identify (i) mechanisms creating spatial patterns when economic agents maximize profit at each site by ignoring the impact of their actions on other sites and (ii) a diffusion induced externality. Pattern formation mechanisms and externalities create a divergence in the spatiotemporal structures emerging under private or social objectives We develop optimal regulation which internalize the spatiotemporal externalities. Our theory is applied to the management and regulation of a semi-arid system. Supporting numerical simulations are also presented.

Keywords: Economic-Ecological Systems, Pattern Formation, Reaction-Diffusion, Diffusion Instability, Spatial Externalities, Regulation.

JEL Classification: Q20, C61, H23.

1 Introduction

The importance of ecosystems to human well-being has been directly related to the provision of services such as food, fuel, fiber, climate regulation or disease control (e.g. Millennium Ecosystem Assessment 2005). An important element of most ecological theories seeking to understand ecosystems is the spatial and temporal structure of ecosystems. Spatial heterogeneity involving spatial structures such as patches or gradients seems to prevail in nature, in contrast to spatial homogeneity, and has a central role in the analysis of issues such as theories of competition, succession, evolution and adaptations, maintenance of species diversity, parasitism, population genetics, population growth, and predator-prey interactions (Legendre and Fortin 1989). Empirical evidence suggests that disturbances in an ecosystem caused by human actions can either increase or decrease landscape heterogeneity depending on the parameter and spatial scale examined (e.g. Mladenoff et al. 1993). This implies that the spatial and temporal structure of ecosystems which are harvested for their services could be important for the human populations benefited by these services.

Arid and semi-arid lands are regarded as a classic example of a system with reaction/diffusion-activator/inhibitor characteristics where striking spatial heterogeneities regarding vegetation patterns emerge (e.g. Tongway and Ludwig 2007). Arid and semi-arid lands, or grazing lands, cover nearly 30% of the earth's land surface and they support both subsistence and commercial grazing.[1] Thus the efficient management of these ecosystems in order to prevent collapse and secure long-term sustainable productivity is an important management issue.

In this context the present paper develops a novel methodological framework which could be helpful in analyzing the formation of spatiotemporal patterns which result from the interactions between decisions taken by economic agents about the management of an ecosystem's resources which generate useful services, and the natural processes which govern the movements of these resources in time and space.

The study of the emergence and the properties of regular spatial or spa-

[1] In Africa and Asia herders use animals for own consumption and then the market. In Africa herders compared to settled farmers produce between 50%-75% of all the milk, beef and mutton produced in the continenet. In Australia, North America and Argentina grazing lands are exploited by large commercial herders (Reid et al. 2008).

tiotemporal patterns which can be found in abundance in nature, such as for example stripes or spots on animal coats, ripples in sandy desserts, vegetation patterns in arid grazing systems or spatial patterns of fish species, has drawn much attention in natural sciences.[2] Reaction-diffusion systems, that is dynamical systems where the state variables interact among each either and at the same time diffuse[3] in space, have been among the main analytical tools for analyzing pattern formation in this context. A classical example is the so-called Turing mechanism (Turing 1952) which provides a framework predicting the emergence of heterogeneous spatial patterns in reaction-diffusion systems. The Turing mechanism explains pattern formation with reference to local instability of a stable spatially homogeneous steady state to spatially heterogeneous perturbations. This mechanism is referred to as diffusion induced instability.

In economics, pattern formation has been associated with the spatial aspects of the economy and concentrates mainly on the study of economic agglomerations at different spatial scales. The analysis of different types of economic agglomerations includes issues such as North-South dualism, patterns of regional growth, the emergence of cities, or the emergence of commercial districts within cities.[4] On the other hand, and given the arguments put forward earlier, the study of the economic management of ecosystems can also be regarded as an area where it seems natural to analyze mechanisms causing spatiotemporal patterns to arise. The analysis of pattern formation in these cases might be useful for the introduction of regulatory policies with spatial characteristics. Although spatial analysis in this context has provided useful insights in areas such as fishery management, spatial pollution, or water pricing, it seems that a unified and systematic analysis of (a) the mechanisms governing the emergence of patterns in ecosystems where spatial diffusion of interacting resources takes place and at the same time resources are managed for economic objectives, and (b) the associated implications for the design of regulatory policies, is still lacking.[5]

[2] See for example Okubo and Levin (2001), Murray (2003), and Hoyle (2006).

[3] In this context diffusion is a proccess through which the microscopic irregular motion of an assemplance of particles such as cells, chemicals, or animals results in a macroscopic regular motion of the group. This classical approach to diffusion implies that diffusion has local or short range effects.

[4] See for example Krugman (1996), Fujita, Krugman and Venables (2001), Fugita and Thisse (2002).

[5] In fisheries the main approach is through metapopulation models, (see for example

Thus in this paper we study the general problem of pattern formation emerging from the interaction of economic objectives, such as the maximization of benefits in a given spatial domain, with natural resources with reaction-diffusion characteristics which evolve in the same spatial domain, and provide (e.g. through harvesting) the benefits that the economic agents seek to maximize. In pursuing this objective we provide what we believe is a novel theoretical treatment for analyzing pattern formation for recursive infinite horizon intertemporal optimization problems where the evolution of state variables is described by a reaction-difusion system. [6]

We structure the interactions between economic management and the natural system in two levels. At the first level we assume that economic agents located at different spatial points (or sites)[7] harvest an ecosystem with reaction-diffusion characteristics, by maximizing private benefits without taking into account the reaction-diffusion characteristics of the natural system. By doing so an externality is generated from agents at each site onto agents at other sites. We call this type of management a private optimization management problem (POMP) and the associated externality a diffusion induced spatial externality (DISE). The natural system's response to a harvesting rule which stems from the POMP could be the emergence of a spatial pattern through the classic Turing mechanism. Alternatively, behavior associated with the POMP could neutralize the Turing mechanism and result in a spatially homogenous situation. Thus private optimizing behavior could create or break patterns. We identify sufficient conditions for pattern formation through the classic Turing mechanism for the POMP. If, in addition, the private agents do not take into account the dynamics of the problem and they act myopically by maximizing current objectives then the

Sanchirico and Wilen, 2005). In pollution management the concept of a spatial distribution of pollution is used and then a two stage optimal control problem is solved (see for example Goetz and Zilberman 2000, Xabadia et al. 2004).

[6] The problem we analyze is a different and more general problem from the one studied by Brock and Xepapadeas (2008) where optimization was constrained by only one state variable. In problems with one state variable, reaction-diffusion features are not possible and thus the uncontrolled single state variable system cannot generate patterns.As shown in Brock and Xepapadeas (2008), patterns in this case are generated by economic forces and are realized in the quantity-price (or state-costate) space-time domain. However, since reaction-diffusion systems which are of dimension higher than one can generate patterns even without the impact of economic forces, they are much more suitable not only for analyzing real ecosystems but also for allowing the study of the interplay between natural forces and economic forces in pattern formation.

[7] This is equivalent to identifying each site with an 'agent'.

usual temporal externality is also emerging since agents ignore the scarcity costs of the harvested resources.[8]

At a second level we examine the problem of a social planner who seeks to internalize both the difusion induced spatial externality and the temporal externality by maximizing discounted benefits over all sites, subject to the constraints imposed by the reaction-diffusion natural system. We call this problem the social optimization management problem (SOMP). We identify conditions for pattern formation associated with the SOMP. However, the emergence of patterns in this case is not the result of the Turing mechanism alone, but the result of interactions between the optimizing behavior and reaction-diffusion. In our recursive optimal control problem which corresponds to the SOMP, pattern formation is associated with the instability to spatially heterogeneous perturbations of a flat optimal steady state with the local saddle point property. We call this type of instability optimal reaction-diffusion instability (ORDI). This instability is identified for a class of problems with two state variables and we believe that it represents a new result.

Comparison of the solutions at the POMP and the SOMP, which could be spatially homogeneous or heterogenous, provides the information about the size and the spatiotemporal structure of spatial and temporal externalities. This information can be used for the design of optimal spatiotemporal regulation, in the sense of determining policy instruments such that a decentralized solution of the POMP under these instruments converges to the SOMP solution. In this context another contribution of our paper is the identification of the reaction-diffusion spatially externality in resource management, and the development of a conceptual framework for designing optimal regulation of reaction-diffusion systems. Given the importance of these systems (such as arid or semi-arid systems) in the real world, such a result provides new insights to regulatory policies.

We apply our theory to a model with the structure of a semi-arid grazing system where we study pattern formation for the POMP, the SOMP, and the design of optimal spatiotemporal regulation.

[8]Private agents could take into account dynamics. However because of the well known strategic effects the resulting open loop or feedback Nash equilibria do not take into account the resources' full scarcity costs. In this case the spatial externality will coexist with a temporal-stategic externality.

2 Pattern Formation and Private Optimization

We consider a reaction-diffusion system consisting of two state variables which react among themselves and diffuse in space. Space is assumed to be a circle of fixed length L. Diffusion is modelled by the classical approach which implies that the concentration of the 'material' (e.g. resource biomass, stock of water in an aquifer) represented by a state variable moves into the spatial domain, from areas of high concentration to areas of low concentration.[9] The reaction diffusion system has activator-inhibitor characteristics. This means that the state variable which is the activator promotes growth in the other variable, while the state variable which is the inhibitor induces depletion of the other state variable. We assume that the evolution of the reaction-difusion system in space-time can be affected by the choice of paths for a vector of control variables which belong to a fixed control set which is independent of the state and control variables. Let $\mathbf{x}(t,z) = (x_1(t,z), x_2(t,z))$ and $\mathbf{u}(t,z) = (u_1(t,z), ..., u_m(t,z))$, $m \geq 1$, denote the vectors of state and control variables respectively, at time $t \in [0,\infty)$ and spatial point $z \in [0,L]$. The reaction-diffusion system can be written as:

$$\frac{\partial x_1(t,z)}{\partial t} = f_1(x_1(t,z), x_2(t,z), \mathbf{u}(t,z)) + D_{x_1}\frac{\partial x_1^2(t,z)}{\partial z} \quad (1)$$

$$\frac{\partial x_2(t,z)}{\partial t} = f_2(x_1(t,z), x_2(t,z), \mathbf{u}(t,z)) + D_{x_2}\frac{\partial x_2^2(t,z)}{\partial z} \quad (2)$$

with boundary conditions

$$x_1(0,z)\ ,\ x_2(0,z) \text{ given} \quad (3)$$

$$\mathbf{x}(t,0) = \mathbf{x}(t,L) = \bar{\mathbf{x}}(t)\,,\ \forall t,\ \text{the space is a circle} \quad (4)$$

The functions $f_i(\mathbf{x}(t,z), \mathbf{u}(t,z)), i = 1,2$ are smooth functions of the state and the control variables and describe the kinetics of the reaction-diffusion system, while the diffusion of the state variables is modelled by the terms $D_{x_1}\frac{\partial x_1^2(t,z)}{\partial z}, D_{x_2}\frac{\partial x_2^2(t,z)}{\partial z}$, with $(D_{x_1}, D_{x_2}) > 0$ being the diffusion coefficients. If $\partial f_1/\partial x_2 > 0$ then x_2 is an activator for x_1, while if $\partial f_2/\partial x_1 < 0$ then x_1 is an inhibitor for x_2.

Assume that an economic agent is located at each spatial point z. Each agent has a benefit function $U(\mathbf{x}(t,z), \mathbf{u}(t,z))$ defined over the state and the control variables. The benefit function is assumed to be increasing and

[9] In more technical terminology the flux of the 'material' at any spatial point is proportional to the gradient of the concentration of the material at this point.

strictly concave in the controls. Each economic agent considers herself/himself to be small in relation to the spatiotemporal evolution of the state variables and thus chooses controls to maximize an objective at each instant of time for the given spatial site, by treating the values of the state variables as exogenous parameters. Thus each agent ignores the impacts of his/her actions on other sites. However, these impacts emerge because of the diffusion of the state variables and this is the source of a diffusion induced spatial externality.

The private controls can be defined in terms of two assumptions which are associated with the type of property rights prevailing in the spatial domain.

If each agent owns enforceable property rights for her/his site the optimal private controls are defined as:

$$u_j^0(z,t) = \arg\max_{u_j} U(\mathbf{x}(t,z), \mathbf{u}(t,z)) \ , j = 1, ..., m \tag{PR}$$

Assume that the optimal private control is interior[10] so $u_j^0(z,t)$ is defined implicitly by the first-order conditions $\frac{\partial U(\mathbf{x}(t,z),\mathbf{u}(t,z))}{\partial u_j} = 0$, $j = 1, ..., m$. Solution of the system of the first-order conditions defines the short-run private optimal controls as feedback rules for given values of the state variables.

$$u_j^0(z,t) = h_j^0(\mathbf{x}(t,z)) \ , j = 1, ..., m \tag{5}$$

The derivatives $\partial u_j^0(z,t)/\partial x_i$ can in principle be determined by using the implicit function theorem as $\frac{\partial \mathbf{u}^0}{\partial \mathbf{x}} = B_{\mathbf{uu}}^{-1} \cdot [-U_{\mathbf{ux}}]$, where $B_{\mathbf{uu}}^{-1}$ is the inverse of the Hessian matrix of the benefit function and $[-U_{\mathbf{ux}}]$ the vector of the second-order mixed partials of the benefit function.[11] It is clear that the impacts of the state variables on the optimal private choices are determined by the assumptions regarding the impacts of the state variables on the benefit function, the so called 'stock effects'.

Under open access, controls are chosen so that rents are dissipated on each site or

$$\hat{\mathbf{u}}(z,t) : U(\mathbf{x}(t,z), \hat{\mathbf{u}}(t,z)) = 0 \text{ for all } s. \tag{OA}$$

[10] Interior solutions can be obtained by assuming $U_{u_j}(0) \to +\infty$, $U_{u_j}(\infty) \to 0$, $j = 1, ..., m$.

[11] To simplify notation we occasionally use subscripts to denote partial derivatives of a function when this does not create confusion.

The open access controls are determined from (OA) in a feedback form as:

$$\hat{u}_j(z,t) = \hat{h}_j(\mathbf{x}(t,z)) \quad , j = 1, ..., m \tag{6}$$

The derivatives $\partial \hat{u}_j(z,t)/\partial x_i$ can in principle be determined as in the (5) case by using the implicit function theorem as: $\frac{\partial \hat{\mathbf{u}}}{\partial \mathbf{x}} = \hat{B}_{\mathbf{u}}^{-1} \cdot [-U_{\mathbf{x}}]$, where $\hat{B}_{\mathbf{u}}^{-1}$ is the inverse of the Jacobian matrix of the benefit function and $-U_{\mathbf{x}}$ is the vector of the partial derivatives with respect to the state variables.

If the private feedback controls (5) or (6) are substituted into (1) and (2), then the reaction-diffusion system determines the spatiotemporal paths of the state variables at the POMP.

To analyze pattern formation we define first a spatially homogeneous or flat steady state (FSS) which is defined from (1) and (2) for $D_{x_1} = D_{x_2} = 0$, as:[12]

$$\begin{aligned} \mathbf{x}^0 = \left(x_1^0, x_2^0\right) &: f_1\left(x_1^0, x_2^0, \mathbf{h}^0\left(\mathbf{x}^0\right)\right) = 0 \\ & \quad f_2\left(x_1^0, x_2^0, \mathbf{h}^0\left(\mathbf{x}^0\right)\right) = 0 \\ \mathbf{h}^0(\mathbf{x}) &= \left(h_1^0(\mathbf{x}), ..., h_m^0(\mathbf{x})\right) \end{aligned} \tag{7}$$

Let $\bar{\mathbf{x}}(t) = \left(x_1(t) - x_1^0, x_2(t) - x_2^0\right)' = (\bar{x}_1(t), \bar{x}_2(t))'$ denote deviations around this FSS and define the linearization

$$\bar{\mathbf{x}}_t(t) = J^P \bar{\mathbf{x}}(t) \ , \ \bar{\mathbf{x}}_t(t) = \begin{pmatrix} \frac{d\bar{x}_1(t)}{dt} \\ \frac{d\bar{x}_2(t)}{dt} \end{pmatrix}, \ J^P = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \tag{8}$$

where the elements of the Jacobian matrix, evaluated at the FSS, are defined as:

$$b_{11} = \frac{\partial f_1}{\partial x_1} + \sum_{j=1}^{m} \frac{\partial f_1}{\partial u_j}\frac{\partial u_j}{\partial x_1} \ , \ b_{12} = \frac{\partial f_1}{\partial x_2} + \sum_{j=1}^{m} \frac{\partial f_1}{\partial u_j}\frac{\partial u_j}{\partial x_2} \tag{9}$$

$$b_{21} = \frac{\partial f_2}{\partial x_1} + \sum_{j=1}^{m} \frac{\partial f_2}{\partial u_j}\frac{\partial u_j}{\partial x_1} \ , \ b_{22} = \frac{\partial f_2}{\partial x_2} + \sum_{j=1}^{m} \frac{\partial f_2}{\partial u_j}\frac{\partial u_j}{\partial x_2} \tag{10}$$

Assume that $\mathrm{tr} J^P = b_{11} + b_{22} < 0$ and $\det J^P = b_{11}b_{22} - b_{12}b_{21} > 0$.[13] This

[12] We use the controls derived from the private optimization problem under full property rights at each site. The approach would have been the same if we were to use the controls derived under open access. The different implications between full property rights and open access will be clarified more when we analyze the semi-arid systems.

[13] This means that both eigenvalues of J^P have negative real parts.

implies that the FSS is locally stable to spatially homogeneous perturbations. To analyze pattern formations we proceed, as for example in Murray (2003), by considering the linearization of the full reaction diffusion system, which is:

$$\bar{\mathbf{x}}_t(t,z) = J^P \bar{\mathbf{x}}(t,z) + D\bar{\mathbf{x}}_{zz}(t,z) \ , \ \bar{\mathbf{x}}_{zz}(t,z) = \begin{pmatrix} \frac{\partial^2 \bar{x}_1(t,z)}{\partial z^2} \\ \frac{\partial^2 \bar{x}_2(t,z)}{\partial z^2} \end{pmatrix}, \ D = \begin{pmatrix} D_{x_1} & 0 \\ 0 & D_{x_2} \end{pmatrix} \tag{11}$$

Spatial patterns emerge if the FSS is unstable to spatially *heterogeneous* perturbations which take the form of spatially varying solutions of (11), defined as:

$$\bar{x}_i(t,z) = \sum_k c_{ik} e^{\sigma t} \cos(kz) \ , \ i = 1,2 \ , k = \frac{2n\pi}{L}, \ n = \pm 1, \pm 2, ... \tag{12}$$

where $k = 2n\pi/L$, and $1/k = L/2n\pi$ is a measure of the wave-like pattern. k is called the *wavenumber* and $1/k$ is proportional to the wavelength ω : $\omega = 2\pi/k = L/n$, while σ is the eigenvalue which determines temporal growth and c_{ik}, $i = 1,2$ are constants determined by initial conditions and the eigenspace of σ. Substituting (12) into (11) and noting that they satisfy circle boundary conditions at $z = 0$ and $z = L$, we obtain the following result:

Theorem 1 *Private optimizing behavior, as implied by choosing controls according to (5) or (6) in the management of a reaction-diffusion system, generates spatial patterns if*

$$\frac{b_{22}D_{x_1} + b_{11}D_{x_2}}{2D_{x_1}D_{x_2}} > 0 \tag{13}$$

$$-\frac{(b_{22}D_{x_1} + b_{11}D_{x_2})^2}{4D_{x_1}D_{x_2}} + \det J^P < 0 \tag{14}$$

For proof see Appendix 1.

If the above conditions are satisfied then, when the spatially *heterogeneous* perturbations are introduced one of the eigenvalues of the linearization matrix of (11) is positive and therefore the steady state $\mathbf{x}^0$ is locally unstable. This implies that a spatially heterogeneous, or patterned, solution emerges and the paths of the state variables in the neighborhood of the FSS can be approximated as:

$$\bar{\mathbf{x}}(t,z) \sim \sum_{n_1}^{n_2} \mathbf{c}_n \exp\left[\sigma\left(k^2\right)\right]\cos(kz) \ , \ k = \frac{2n\pi}{L} \tag{15}$$

where the vector $\mathbf{c}_n$ is determined by initial conditions at date $t = 0$ and $\sigma\left(k^2\right) > 0$ for $k^2 \in \left(k_1^2, k_2^2\right)$. n_1 is the smallest integer greater than or equal to $Lk_1/2\pi$ and n_2 is the largest integer less than or equal to $Lk_2/2\pi$, and the wavenumbers k_1 and k_2 are such that the so-called dispersion relationship satisfies

$$\phi\left(k^2\right) = D_{x_1}D_{x_2}k^4 - \left(b_{22}D_{x_1} + b_{11}D_{x_2}\right)k^2 + \det J^P < 0 \tag{16}$$

for $k^2 \in \left(k_1^2, k_2^2\right)$. The dispersion relationship is central to this type of analysis. A negative dispersion relationship implies that when solutions (12) are substituted into (11) then, as shown in Appendix 1, one of the eigenvalues of the system's Jacobian matrix becomes positive for given diffusion coefficients and all $k^2 \in \left(k_1^2, k_2^2\right)$. As time t increases, the dominant contribution to the spatiotemporal path $\bar{\mathbf{x}}(t,z)$ approximated by (15) will come from unstable modes for which the eigenvalue $\sigma\left(k^2\right)$ of (11) that determines temporal growth is positive. Modes with $\sigma\left(k^2\right) < 0$ will fade away in influence as t increases and thus a spatial pattern is generated, provided that the length L of space allows the existence of these unstable modes. As noted in Murray (2003) if the exponentially growing solution (15) were valid for all time it would imply that $\mathbf{x}(t,z) \to \infty$ as $t \to \infty$. It is hypothesized that the kinetics of the system (1), (2) bound the solution $\mathbf{x}(t,z)$ which eventually settles to a spatial pattern. A spatially heterogeneous steady state (HSS) can be obtained as the solution of the second-order system in the spatial variable z.

$$\begin{aligned} 0 &= f_1\left(\mathbf{x}(z), \mathbf{h}^0(z)\right) + D_{x_1}\frac{\partial^2 x_1(z)}{\partial z^2} && (17) \\ 0 &= f_2\left(\mathbf{x}(z), \mathbf{h}^0(z)\right) + D_{x_2}\frac{\partial^2 x_2(z)}{\partial z^2} && (18) \end{aligned}$$

This second order system (17), (18) can be either solved numerically or be transformed to a (4×4) first-order system by the transformation $\mathbf{X} = \frac{\partial \mathbf{x}}{\partial z}$ and then solved given appropriate boundary conditions.

It is important to note the impact of private optimizing decisions in the creation or destruction of spatial patterns through the impact of terms $\xi_{ij} = \sum_{j=1}^{m} \frac{\partial f_i}{\partial u_j} \frac{\partial u_j}{\partial x_i}$, $i = 1, 2$ which embody the influence on the state variables resulting from profit-maximizing choices of the controls, when diffusion of the state variables and interdependences among them are ignored when these controls are chosen. If patterns emerge in this set up, their creation is a result of the Turing mechanism. Patterns can of course emerge through the same mechanism when the system is uncontrolled. This is the case of 'undisturbed Nature' where the conditions for the emergence of patterns are similar to (13) and (14) but the terms b_{ij} do not contain the impacts ξ_{ij} which come from the private optimizing behavior. These arguments suggest that spatial patterns formed by a reaction diffusion system in the state of undisturbed Nature will be in general different from those emerging when the system is controlled under private optimization objectives. Given this result it might be of importance to analyze the type of spatiotemporal patterns emerging from a behavior that seeks to attain the socially optimal management of the reaction diffusion system. This is the problem examined in the next section.

3 Pattern Formation and Social Optimization

Using the same structure of the reaction diffusion systems described by (1)-(4), we introduce a social planner who has as objective the maximization of discounted benefits over the whole spatial domain subject to the constraints imposed by (1)-(4). By explicitly taking into account these constraints the social planner internalizes spatial and temporal externalities which were not taken into account at the private optimum. The problem of the social planner can be stated as:

$$\max_{\{\mathbf{u}(z,t)\}} \int_0^\infty \int_0^L e^{-\rho t} \left[U\left(\mathbf{x}\left(t,z\right), \mathbf{u}\left(t,z\right)\right) \right] dz dt \tag{19}$$
$$\text{subject to } (1) - (4)$$

To use the maximum principle under spatial diffusion for problem (19), we follow Derzko et al. (1984, pp. 95-96) and Brock and Xepapadeas (2008), and we introduce the Hamiltonian function:

$$\mathcal{H}\left(\mathbf{x}\left(t,z\right), \mathbf{u}\left(t,z\right), \mathbf{p}\left(t,z\right)\right) = U\left(\mathbf{x}\left(t,z\right), \mathbf{u}\left(t,z\right)\right) \tag{20}$$
$$+ \sum_{i=1,2} p_i\left(t,z\right) \left[f_i\left(\mathbf{x}\left(t,z\right), \mathbf{u}\left(t,z\right)\right) + D_{x_i} \frac{\partial x_i^2\left(t,z\right)}{\partial z^2} \right] \tag{21}$$

which is a generalization of the 'flat' Hamiltonian function

$$H = U(\mathbf{x}, \mathbf{u}) + \sum_{i=1,2} p_i f_i(\mathbf{x}, \mathbf{u}) \tag{22}$$

where $\mathbf{p}(t,z) = (p_1(t,z), p_2(t,z))$ is the vector of the costate variables. The first-order conditions for the optimal control vector $\mathbf{u}^*(t,z)$ imply $u_j^*(t,z) = \arg\max_{u_j} \mathcal{H}(\mathbf{x}(t,z), \mathbf{u}(t,z), \mathbf{p}(t,z))$. For interior solutions $u_j^*(t,z)$ is defined by

$$\frac{\partial \mathcal{H}}{\partial u_j} = \frac{\partial U(\mathbf{x}(t,z), \mathbf{u}(t,z))}{\partial u_j} + \sum_{i=1,2} p_i(t,z) \frac{\partial f_i(\mathbf{x}(t,z), \mathbf{u}(t,z))}{\partial u_j} = 0 \, , j = 1, ...m \tag{23}$$

Solving the above system,[14] optimal short-run controls are defined in terms of the state and the costate variable

$$u_j^*(t,z) = g_j^*(\mathbf{x}(t,z), \mathbf{p}(t,z)) \ \ , j = 1, ...m \tag{24}$$

The costate variables need to satisfy:

$$\frac{\partial p_i(t,z)}{\partial t} = \rho - H_{x_i}(\mathbf{x}(t,z), \mathbf{p}(t,z), \mathbf{g}^*\mathbf{x}(t,z)) - D_{x_i} \frac{\partial^2 p_i(t,z)}{\partial z^2} \, , i = 1, 2 \tag{25}$$

where $\mathbf{g}^*(\mathbf{x}(t,z), \mathbf{p}(t,z))$ is the vector of the optimal control functions defined by (24). Finally the following temporal and spatial transversality conditions should be satisfied at the optimum:

$$\lim_{T \to \infty} e^{-\rho T} \int_0^L p_i(T,z) x_i(T,z) dz = 0 \ , i = 1, 2 \tag{26}$$

$$p_i(t,0) = p_i(t,L) \ \ , i = 1, 2 \tag{27}$$

The system of (1) and (2) with $\mathbf{u}$ replaced by the optimal short-run controls $\mathbf{g}^*(\mathbf{x}(t,z), \mathbf{p}(t,z))$ and the system of (25) constitute a system of four partial differential equations. This is the modified Hamiltonian system (MHS), which along with the initial conditions (3), (4) and the transversality conditions (26), (27) determine the spatiotemporal evolution of the state and costate variables along the socially optimal path.

To analyze pattern formation at the social optimum we again examine

[14] The Hamiltonian needs to be concave in the state and the control variables.

the stability of the flat optimal steady state (FOSS) of the MHS to spatially heterogeneous perturbations. The MHS can be written in a compact way, using subscripts to denote derivatives in order to simplify notation, as:

$$
\begin{aligned}
x_{1t} &= H_{p_1} + D_{x_1} x_{1zz} && (28)\\
x_{2t} &= H_{p_2} + D_{x_2} x_{2zz} && (29)\\
p_{1t} &= \rho p_1 - H_{x_1} - D_{x_1} p_{1zz} && (30)\\
p_{2t} &= \rho p_2 - H_{x_2} - D_{x_2} p_{2zz} && (31)
\end{aligned}
$$

A FOSS is defined, from the MHS (28)-(31), as a quadruple $(x_1^*, x_2^*, p_1^*, p_2^*)$: $x_{1t} = x_{2t} = p_{1t} = p_{2t} = 0$ for $D_{x_1} = D_{x_2} = 0$. It is known from the work of Kurz (1968) that such a FOSS will be either unstable or will have the saddle point property. Furthermore as shown by Dockner (1985), by explicit calculation of the eigenvalues of the (4×4) Jacobian matrix (28)-(31) at the FOSS, a FOSS could: (i) be complete unstable with all eigenvalues having positive real parts, (ii) be unstable, except for a one-dimensional manifold, with three eigenvalues with positive real parts and one with a negative real part, or (iii) have the saddle point property with two eigenvalues with positive real parts and two eigenvalues with negative real parts. In the infinite horizon optimal control problems with n state variables studied in economics the saddle point property is of particular interest. The combination of the saddle point property with a transversality condition at infinity, allows us to set the n constants, which correspond to eigenvalues with positive real parts of the solution of the $2n$-dimensional MHS, equal to zero, and then determine the remaining constants, so that for any initial state vector in the neighborhood of the FOSS, there is an initial costate vector such that the system converges on the n-dimensional stable manifold to the FOSS.

Thus to study pattern formation we will concentrate on the case of a FOSS with the saddle point property. Let

$$
\begin{aligned}
\bar{\mathbf{x}}(t,z) &= \left(x_1(t,z) - x_1^*, x_2(t,z) - x_2^*\right)' = \left(\bar{x}_1(t,z), \bar{x}_2(t,z)\right)'\\
\bar{\mathbf{p}}(t,z) &= \left(p_1(t,z) - p_1^*, p_2(t,z) - p_2^*\right)' = \left(\bar{p}_1(t,z), \bar{p}_2(t,z)\right)'
\end{aligned}
$$

denote deviations from this FOSS, and define the linearization of the MHS

(28)-(31)

$$\begin{pmatrix} \bar{\mathbf{x}}_t(t,z) \\ \bar{\mathbf{p}}_t(t,z) \end{pmatrix} = J^0 \begin{pmatrix} \bar{\mathbf{x}}(t,z) \\ \bar{\mathbf{p}}(t,z) \end{pmatrix} + D \begin{pmatrix} \bar{\mathbf{x}}_{zz}(t,z) \\ \bar{\mathbf{p}}_{zz}(t,z) \end{pmatrix} \tag{32}$$

$$J^0 = \begin{pmatrix} H_{\mathbf{px}} & H_{\mathbf{pp}} \\ -H_{\mathbf{xx}} & \rho I_2 - H_{\mathbf{xp}} \end{pmatrix}, \; D = \begin{pmatrix} D_{x_1} & 0 \\ 0 & D_{x_2} \end{pmatrix} \tag{33}$$

where $H_{\mathbf{pp}}, H_{\mathbf{xx}}, H_{\mathbf{px}} = H_{\mathbf{xp}}$ are (2×2) matrices of second derivatives of the Hamiltonian calculated from (28)-(31) with $\mathbf{u} = \mathbf{g}^*(\mathbf{x}, \mathbf{p})$, I_2 is the (2×2) identity matrix, and J^0 is the Jacobian of the MHS of the spatially homogeneous, that is $D_{x_1} = D_{x_2} = 0$, system. All derivatives are evaluated at the FOSS.

Consider spatially heterogeneous perturbations of the FOSS of the form

$$\bar{x}_i(t,z) = \sum_k c_{ik}^x e^{\sigma t} \cos(kz), \; \bar{p}_i(t,z) = \sum_k c_{ik}^p e^{\sigma t} \cos(kz) \;, k = \frac{2n\pi}{L}, \; n = \pm 1, \pm 2, ... \tag{34}$$

which are substituted as trial solutions into the linearization (32) and define the following:

$$K_i = \begin{vmatrix} H_{p_i x_i} - D_{x_i} k^2 & H_{p_i p_i} \\ -H_{x_i x_i} & \rho - H_{x_i p_i} + D_{x_i} k^2 \end{vmatrix}, \; i = 1,2 \tag{35}$$

$$K_3 = \begin{vmatrix} H_{p_1 x_2} & H_{p_1 p_2} \\ -H_{x_1 x_2} & H_{x_1 p_2} \end{vmatrix}, \; K\left(k^2\right) = K_1 + K_2 + 2K_3 \tag{36}$$

$$K^0 = K_1 + K_2 + 2K_3 \quad \text{for} \;\; D_{x_1} = D_{x_2} = 0 \;, i = 1,2 \tag{37}$$

$$K^0 = \sum_{i=1,2} \left[\left(\rho - H_{x_i p_i}\right) H_{p_i x_i} + H_{x_i x_i} H_{p_i p_i} \right] \tag{38}$$

$$+ \left[-H_{p_1 x_2} H_{x_1 p_2} + H_{x_1 x_2} H_{p_1 p_2} \right] \tag{39}$$

$$K\left(k^2\right) = -\left(D_{x_1}^2 + D_{x_2}^2\right) k^4 + \left[\sum_{i=1,2} D_{x_i} (2H_{x_i p_i} - \rho) \right] k^2 + K^0 \tag{40}$$

$$J^S = \begin{pmatrix} H_{\mathbf{px}} - Dk^2 I_2 & H_{\mathbf{pp}} \\ -H_{\mathbf{xx}} & \rho I_2 - H_{\mathbf{xp}} + Dk^2 \end{pmatrix} = J^S\left(k^2\right) \tag{41}$$

Then the following theorem can be stated.

Theorem 2 *Assume that for problem (19) with $D_{x_1} = D_{x_2} = 0$, the FOSS*

$(x_1^*, x_2^*, p_1^*, p_2^*)$ *associated with the Jacobian matrix* J^0 *has the local saddle point property with either two positive and two negative real roots, or with complex roots with two of them having negative real parts. Then there is a* $(D_{x_1}, D_{x_2}) > 0$ *and wave numbers* $k \in (k_1, k_2) > 0$ *such that, if:* ***(a)***

$$\frac{\left[\sum_{i=1,2} D_{x_i}(2H_{x_i p_i} - \rho)\right]}{2\left(D_{x_1}^2 + D_{x_2}^2\right)} > 0 \tag{42}$$

$$\frac{\left[\sum_{i=1,2} D_{x_i}(2H_{x_i p_i} - \rho)\right]^2}{4\left(D_{x_1}^2 + D_{x_2}^2\right)} + K^0 > 0 \tag{43}$$

$$0 < \det J^S\left(k^2\right) \leq (K/2)^2$$

then all the eigenvalues of the linearization matrix (41) of the system (32) are real and positive. ***(b)***

$$\det J^S\left(k^2\right) < 0 \tag{44}$$

then the linearization matrix (41) of the system (32) has one negative real eigenvalue, while all the other eigenvalues have positive real parts. ***(c)***

$$\begin{aligned} K^2 - 4\det J^S\left(k^2\right) &< 0 \qquad (45) \\ \det J^S\left(k^2\right) &< (K/2)^2 + \rho^2 (K/2) \end{aligned}$$

then all the eigenvalues of the linearization matrix (41) of the system (32) are complex with positive real parts. In all cases above the optimal dynamics associated with the reaction-diffusion system are unstable in the neighborhood of the FOSS, in the time-space domain.

For proof see Appendix 2.

Theorem 2 states that when spatial perturbations are introduced, then if any of the (a),(b) or (c) are satisfied, the two negative eigenvalues associated with the linearization matrix J^0 of the FOSS become either both positive (cases a and c), or one remains negative while the other are positive. In all cases a patterned solution emerges for the state and costate paths in the neighborhood of the FOSS.

In all three cases the corresponding dispersion relationship is central in understanding the emergence of spatial instability. The saddle point property of the FOSS means that there exists a two-dimensional manifold such that for any initial values $\mathbf{x}(0)$ for the state variables in the neighborhood of the FOSS there exist initial values $\mathbf{p}(0)$ for the costate variables, such that the system paths $(\mathbf{x}(t), \mathbf{p}(t))$ converge to the FOSS as $t \to \infty$. Furthermore, if the curvature condition on the maximized Hamiltonian is satisfied, then this manifold is globally stable (Brock and Scheinkman 1976). In the neighborhood of the FOSS the nonlinear stable manifold can be approximated by the tangent linear subspace which is spanned by the negative eigenvalues of the linearization around the FOSS.

When conditions (a) of theorem 2 are satisfied, then the dispersion relationship (40) becomes positive for a range of positive wave numbers. Denote by $\left(\sigma_3\left(k^2\right), \sigma_4\left(k^2\right)\right) > 0, k^2 \in \left(k_1^2, k_2^2\right)$ the eigenvalues that turn positive under spatial perturbation, then the patterned state and costate paths can be approximated as:

$$\begin{pmatrix} \bar{\mathbf{x}}(t,z) \\ \bar{\mathbf{p}}(t,z) \end{pmatrix} \sim \sum_{n_1}^{n_2} \mathbf{c}_{3n} \exp\left[\sigma_3\left(k^2\right)\right] \cos(kz) + \sum_{n_1}^{n_2} \mathbf{c}_{4n} \exp\left[\sigma_4\left(k^2\right)\right] \cos(kz)\ ,\ k = \frac{2n\pi}{L} \tag{46}$$

It should be noted that the two constants which correspond to eigenvalues σ_1, σ_2 with positive real parts should be set equal to zero, so that the use of the temporal transversality condition at infinity will allow for any initial state $\mathbf{x}$ to choose initial costates $\mathbf{p}$. This choice will set the system on the *spatially heterogeneous - spatiotemporally unstable*, 'optimal' manifold. The x components of the vectors $(\mathbf{c}_{3n}, \mathbf{c}_{4n})$ are determined by initial values on $\mathbf{x}$, while the p components are determined by the eigenspace spanned by the two eigenvalues $\sigma_{3,4}$. Furthermore, as in the previous section, n_1 is the smallest integer greater than or equal to $Lk_1/2\pi$ and n_2 is the largest integer less than or equal to $Lk_2/2\pi$, and the wavenumbers k_1 and k_2 are such that for $k^2 \in \left(k_1^2, k_2^2\right)$ the dispersion relationship (40) is positive. The length L of space should be sufficient to allow the existence of these unstable modes.

In case (b) the dispersion relationship $\det J^S\left(k^2\right)$, with $J^S\left(k^2\right)$ given by (41), becomes negative for a range of positive wave numbers. Let $\sigma_3\left(k^2\right) < 0, \sigma_4\left(k^2\right) > 0$, $k^2 \in \left(k_1^2, k_2^2\right)$ then the patterned state and costate paths can be approximated again by (46) with $\sigma_3 < 0$, $\sigma_4 > 0$. The fact that one real negative eigenvalue exists, does not mean that the system can be controlled on the *one-dimensional* stable manifold so that spatial patterns will die out. This is because if the constants of the solution of the Hamiltonian system associated with the three negative eigenvalues are set equal to zero, then there is one constant to be determined and two independent initial values for the state variable. In this case a constant that satisfies both initial conditions cannot be determined. To put it another way, for any two independent - that is no proportional to each other - initial states and a transversality condition at infinity, an initial costate vector cannot be chosen such that the system converges on the *one dimensional* stable manifold to the FOSS. In this case initial states on $\mathbf{x}$ and the eigenspace associated with the eigenvalues $\sigma_3\left(k^2\right) < 0, \sigma_4\left(k^2\right) > 0$ can be used to determine $\mathbf{c}_{3n}$ and $\mathbf{c}_{4n}$. As time increases patterns associated with $\sigma_3\left(k^2\right)$ will die out while patterns associated with $\sigma_4\left(k^2\right)$ will grow. Thus optimal dynamics are unstable and a spatial pattern emerges.

In case (c) the dispersion relationship is in the form of the system of inequalities (45). For wavenumbers $k^2 \in \left(k_1^2, k_2^2\right)$ such that (45) are satisfied, the complex eigenvalues with positive real parts will be $\sigma_{3,4}^c\left(k^2\right) = u\left(k^2\right) \pm iv\left(k^2\right), u\left(k^2\right) > 0$. The patterned state and costate paths can be approximated as in (46) with σ replaced by σ^c. In this case we have temporal fluctuations as the spatial patterns are emerging.

Patterned spatiotemporal paths as described above grow exponentially around the FOSS; this, however, cannot be valid for all t, since then exponential growth would imply that $(\mathbf{x}, \mathbf{p}) \to \infty$ at $t \to \infty$. However, the kinetics of the Hamiltonian system (28)-(31) and the transversality condition at infinity (26) should bound the solution. This suggests that the growing solution of the MHS might settle to a certain spatial pattern as $t \to \infty$[15] and a spatially Heterogeneous Optimal Steady-State solution (HOSS) for the optimally controlled system will emerge. This HOSS will satisfy the system of second-order differential equations in the space variable z, defined by (28)-(31) for $x_{1t} = x_{2t} = p_{1t} = p_{2t} = 0$, or,

$$\mathbf{0} = H_{\mathbf{p}} + D\mathbf{x}'_{zz} \tag{47}$$

$$\mathbf{0} = \rho\mathbf{p}' - H_{\mathbf{x}} - D\mathbf{p}'_{zz} \tag{48}$$

This second order (4×4) system can either be solved numerically with appropriate boundary conditions, or can be transformed to an (8×8) first-

[15] See Brock and Xepapadeas (2008) for this argument.

order system by the transformation $\mathbf{X}=\frac{\partial \mathbf{x}}{\partial z}$, $\mathbf{P}=\frac{\partial \mathbf{p}}{\partial z}$ and then solved given the spatial boundary conditions on the circle.

To obtain more insights into the structure of spatial instability implied by theorem 2 we consider briefly the problem of the optimal linear regulator under spatial diffusion.

3.1 Optimal Linear Regulator under Spatial Diffusion

The problem of a deterministic optimal linear regulator under spatial diffusion (LR-SD) can be defined as:[16]

$$\max_{\{\mathbf{u}(t,z)\}} -\int_0^\infty \int_0^L e^{-\rho t}\left[\mathbf{x}'R\mathbf{x}+\mathbf{u}'Q\mathbf{u}\right] dz dt$$

subject to:

$$\mathbf{x}_t = -A\mathbf{x}+B\mathbf{u}+D\mathbf{x}_{zz}$$

$$\mathbf{x}(0,z) \text{ given },\ \mathbf{x}(t,0)=\mathbf{x}(t,L)=\bar{\mathbf{x}}(t),\ \forall t$$

where R,Q are (2×2) positive definite matrices, A,B are given (2×2) matrices and D is the matrix of the difussion coefficients. The maximum principle under spatial diffusion implies

$$H = -\left(\mathbf{x}'R\mathbf{x}+\mathbf{u}'Q\mathbf{u}\right)+\boldsymbol{\lambda}'\left[-A\mathbf{x}+B\mathbf{u}+D\mathbf{x}_{zz}\right] \tag{49}$$

$$\mathbf{u}^* \text{maximizes } H, \text{ or } \mathbf{u}^*=2Q^{-1}B'\boldsymbol{\lambda} \tag{50}$$

$$\mathbf{x}_t = H_{\mathbf{p}}+D\mathbf{x}_{zz}=-A\mathbf{x}+2BQ^{-1}B'\boldsymbol{\lambda}+D\mathbf{x}_{zz} \tag{51}$$

$$\boldsymbol{\lambda}_t=\rho I_2 - H_{\mathbf{x}}-D\boldsymbol{\lambda}_{zz}=\left(\rho I_2+A'\right)\boldsymbol{\lambda}+2R\mathbf{x}-D\boldsymbol{\lambda}_{zz} \tag{52}$$

Figure 1 presents a dispersion relationship associated with part (b) of theorem 2 derived from a numerical simulation of the LR-SD problem.[17]

From the Hamiltonian system $(51),(52)$, the FOSS obtained for $D_{x_1}=D_{x_2}=0$, is a saddle point with complex roots. For $k^2\in(0.831,1.702)$, the Jacobian of the system $(51),(52)$ under the spatial perturbation, which is the dispersion relationship, has two complex eigenvalues with positive real parts and two real eigenvalues, one positive and one negative. This is part

[16] Cross products between $\mathbf{x}$ and $\mathbf{u}$ can be eliminated by suitable transformations of $\mathbf{x}$ and $\mathbf{u}$ (Hansen and Sargent 2007).

[17] The parameters are $R_{11}=0.5, R_{12}=-0.07$; $R_{21}=0.07, R_{22}=0.5, Q_{11}=-0.1, Q_{12}=0.08$, $Q_{21}=-0.08, Q_{22}=-0.1, A_{11}=-0.1, A_{12}=0, A_{21}0$,
$A_{22}=-0.1, B_{11}=0.01, B_{12}=-0.01, B_{21}=0.01$; $B_{22}=0.01, \rho=0.01$, $D_{x_1}=0.1$, $D_{x_2}=1$.

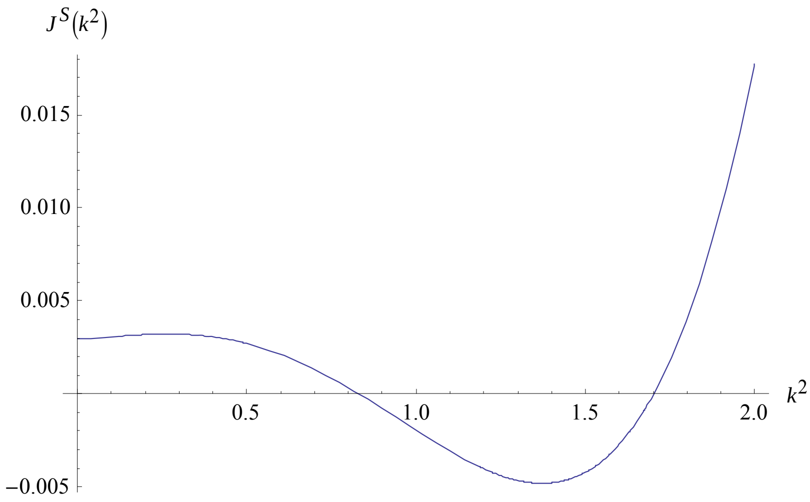

Figure 1: Dispersion relationship for the linear regulator. $D_{x_1} = 0.1$, $D_{x_2} = 1$

(b) of theorem 2 implying that a pattern emerges in the neighborhood of the FOSS.

4 Optimal Diffusion Induced Instability, Externalities and Regulation

As the conditions of Theorem 1 suggest, the spatial patterning for the POMP is driven by a Turing type mechanism since the feedback nature of the optimal control given by (5) is incorporated into the activator-inhibitor structure of the model, which along with diffusion drives the formation or the crashing of spatial patterns. Thus in this case the economic behavior does not change the basic structure of the reaction-diffusion/inhibitor-activator model. At the SOMP however spatial patterning is determined by the structure of the MHS (28)-(31), which retains inhibitor-activator characteristics for the subsystem related to the state variables, but the structure of the whole system is different. In the MHS the state variables have positive diffusion coefficients, while the costate variables have negative diffusion coefficients. This indicates that quantity variables (the state variables) move from high concentration to low concentration, as in classical diffusion, while the price variables (costate variables) move the opposite way as is natural in economic systems. Under this structure, pattern formation is governed by the curvature of the Hamiltonian function and the size of the discount rate as indicated by the conditions of theorem 2. Spatial instability in this case is emerging because the optimized system attains a higher value at the optimal spatially heterogenous state relative to the spatially homogeneous state. In this sense the spatial instability associated with the SOMP is an optimal diffusion-induced instability (ODI).

The different sources of spatial instability are expected to generate in general different spatial patterns. By comparing the solutions obtained in the two previous sections it is clear that the spatiotemporal patterns emerging at the 'undisturbed Nature' - the POMP and the SOMP - will in general be different from each other. It should be noted that one system, say the POMP, could exhibit a spatially heterogeneous pattern while the other, the SOMP, a spatially homogenous pattern, since spatial patterns can be formed but can also be eliminated by the change in parameters and the dimensionality induced by alternative behavioral assumptions regarding the choices of the control variables. On the other hand the diffusion induced spatial externality is present at the reaction-diffusion system, because agents ignore the impact of their decisions on the concentration of the state variables which are located on the sites of other agents. If we assume that in human managed systems the desired solution is the one associated with the SOMP, then the need for regulation arises.

Some broad features of regulatory schemes become clearer if we consider the system at a temporal steady state (which could be spatially homogeneous or heterogeneous), in which case only diffusion, and not temporal growth, affects the concentration of the state variables. The first-order conditions for selecting controls for the POMP and SOMP at a temporal steady state, are respectively

$$\frac{\partial U\left(\mathbf{x}\left(z\right),\mathbf{u}\left(z\right)\right)}{\partial u_j}=0\ ,j=1,...,m \tag{53}$$

$$\frac{\partial U\left(\mathbf{x}\left(z\right),\mathbf{u}\left(z\right)\right)}{\partial u_j}+\sum_{i=1,2}p_i\left(z\right)\frac{\partial f_i\left(\mathbf{x}\left(z\right),\mathbf{u}\left(z\right)\right)}{\partial u_j}=0 \tag{54}$$

The fact that the shadow value of the state variables *at each site*, reflected in the costate variables $p_i\left(z\right)$, is not taken into account at the POMP creates the divergence between the POMP and the SOMP. Thus a decentralized regulatory scheme should be based on introducing these spatially heterogeneous shadow values into the private agent's maximization problem. Therefore regulation will be characterized by spatial heterogeneity.[18] To make a simple case assume that the functions f_i are separable in the states and the controls, and linear in the controls. Then (54) will be of the general form

$$\frac{\partial U(\mathbf{x},\mathbf{u})}{\partial u_j}+\gamma_j^1 p_1(z)+\gamma_j^2 p_1(z)=0\ ,j=1,...,m$$

where $\gamma_j^i, i=1,2$ are constants. Let $p_i^*(z)$ be optimal steady state spatial paths for the costate variables obtained as a solution of (47), (48). Then if site dependent taxes $\tau(z)=\gamma_j^i p_i^*(z)$ per unit of $u_j(z)$ are introduced the private optimization problem will be

$$\max_{\mathbf{u}(t,z)} U(\mathbf{x}(z),\mathbf{u}(z))-\sum_{j=1}^{m}\sum_{i=1,2}\left[\gamma_j^i p_i^*(z)\right]u_j(t,z)$$

and the first order conditions will be identical to (54). Thus the conditions for choosing controls optimally are the same for the SOMP and for the regulated POMP. If in the same set up we consider the full spatiotemporal paths, where both spatial and temporal externalities are present at the unregulated POMP, then the taxes will be site and time dependent and will have the form $\tau(t,z)=\gamma_j^i p_i^*(t,z)$ per unit of $u_j(t,z)$, where $p_i^*(t,z)$ are the optimal spatiotemporal paths for the costate variables obtained as a solution of (19).

In a similar way quantity instruments can be introduced. Let the spatial paths for the optimal controls which correspond to the SOMP be $(u_1^*(z),u_2^*(z))$. These paths can be set as quantity constraints to the private agents. If the constraints are binding then control paths between the SOMP and the regulated POMP are identical. The quantity constraints can be implemented either by command and control methods or by setting up markets for tradable quotas across markets.

The structure of regulation will became clearer in the application which follows.

[18] This concept is similar to the time dependent zonal taxes, zonal permits, and zonal standards discussed by Goetz and Zilberman 2000

5 Management of Arid or Semi-arid Grazing Systems

Arid or semi-arid grazing systems are a typical example of natural systems where pattern formation regarding vegetation emerges in the set up discussed above. A semi-arid system can be described in terms of spatiotemporal dynamics of three state variables: surface water, soil water, and plant biomass (e.g. HilleRisLambers et al. 2001, van de Koppel et al. 2002). We assume, in order to simplify the structure of our model without loss of generality with respect to the issues that we want to explore, that space is a circle, in contrast to the two-dimensional space of the relevant literature, and that surface water is fixed by rainfall and uniformly distributed along the circle. To introduce economic behavior in the management of these systems we assume that plant biomass is consumed in the process of producing cattle products. Cattle products are produced by a conventional production function with two inputs, plant biomass and grazing effort. The model of a semi-arid grazing system can be written as:

$$
\begin{aligned}
&P_t(t,z) = G(W(t,z), P(t,z)) - b(P) - TH(t,z) + D_P P_{zz}(t,z) \qquad (55)\\
&W_t(t,z) = F(P(t,z), R) - V(W(t,z), P(t,z)) - r_W W(t,z) + D_W W_{zz}(t,z)\\
&P(0,z),\ W(0,z) \text{ given}\\
&P(t,0) = P(t,L) = \bar{P}(t),\ \forall t\\
&W(t,0) = W(t,L) = \bar{W}(t),\ \forall t
\end{aligned}
$$

where $P(t,z)$ which denotes plant density or plant biomass, and $W(t,z)$ which denotes soil water at time $t \in [0,\infty)$ and site $z \in [0,L]$, are the state variables. Rainfall R is fixed and $TH(t,z)$ denotes total harvesting of the plant biomass through grazing. Thus total biomass consumption at time t and site z is given by $TH(t,z)$. The function $G(W,P)$ describes plant growth which is assumed to be increasing both in soil water and plant density, $b(P)$ describes plant senescence, $F(P,R)$ describes water infiltration as an increasing function of plant biomass and rainfall, $V(W,P)$ describes water uptake by plants as an increasing function of soil water content and plant density, r_W is the specific rate of water loss due to evaporation and percolation, and D_P and D_W are diffusion coefficients for plant biomass (plant dispersal) and soil water.[19]

To make the procedure more precise and suitable for numerical simulations, we follow van de Koppel et al. (2002) in choosing specific functional forms.

$$\begin{aligned} G(W,P) &= gWP^{\eta+1}\ ,\ b(P) = d(1+\delta P)P, \quad \eta \geq 0, d, \delta > 0 \\ F(P,R) &= (\beta + \zeta P)R\ ,\ V(W,P) = uWP\ ,\ \beta, \zeta, u > 0 \end{aligned}$$

In the system described by (55) grazing could occur naturally by herbivores at a state where the given semi-arid system is not disturbed by human actions. In this case harvesting at the state of 'undisturbed nature' can be approximated as

$$TH^{un}(t,z) = c_{avg}P(t,z) \tag{56}$$

where c_{avg} can be regarded as a constant defined as $c(HERB_{avg})$ where $HERB_{avg}$ is average local herbivore abundance and c is a consumption constant.

5.1 Pattern Formation in a semi-arid System under Private Objectives

We consider now the case where the system is managed by economic agents located at each site, whose objective is maximization of private profit from cattle products. To obtain the cattle products the economic agents harvest plant biomass by exercising costly grazing or harvesting effort. Total harvesting is determined by a Cobb-Douglas function with constant returns to scale in plant biomass and grazing effort, defined as:

$$TH(t,z) = [P(t,z)]^{\alpha}[E(t,z)]^{1-\alpha} \tag{57}$$

We assume that the cost per unit of grazing effort is constant and independent of site, that cattle products are proportional to harvesting and that

[19] Extension to a two-dimensional (z_1, z_2) space is straightforward. In the two-dimensional space the diffusion terms are defined as $D_P(P_{z_1z_1} + P_{z_1z_1})$ and $D_W(W_{z_1z_1} + W_{z_1z_1})$. Spatial heterogeneous perturbation in this case will take the general form $\exp(\sigma t)[\cos(k_1z_1)\cos(k_2z_2)]$.

the price of cattle products is exogenous to the system.[20] Then the grazing effort which maximizes profits from cattle products maximizes also profits from harvesting, with the price per unit harvest expressed in terms of the exogenous price per unit of cattle products. Profits from harvesting are defined as $p\left[P\left(t,z\right)\right]^{\alpha}\left[E\left(t,z\right)\right]^{1-\alpha}-cE\left(t,z\right).$

Using the results of section 2, effort and harvesting at the POMP are determined, for the case of full property rights, as:

$$E^{0}\left(t,z\right)=\gamma P\left(t,z\right),\ TH^{0}\left(t,z\right)=AP\left(t,z\right),\gamma=\left(\frac{c}{p\left(1-a\right)}\right)^{-\frac{1}{a}},A=\gamma^{1-a} \tag{58}$$

For the case of open access, effort and harvesting are determined as:

$$\hat{E}\left(t,z\right)=\hat{\gamma}P\left(t,z\right),\ T\hat{H}\left(t,z\right)=\hat{A}P\left(t,z\right),\hat{\gamma}=\left(\frac{c}{p}\right)^{-\frac{1}{a}},\hat{A}=\hat{\gamma}^{1-a} \tag{59}$$

Thus in both cases harvesting at each site is proportional to plant biomass at the same site. If we make the usual assumptions that $p>c>0$ and $0<a<1$, then $\hat{A}>A$ and, as expected, for any given level of plant biomass harvesting under open access is higher relative to the profit maximizing harvesting under enforceable property rights at each site. Substituting $H^{0}\left(t,z\right)$[21] into (55) we obtain the dynamic system that describes the spatiotemporal evolution of the plant biomass and the soil water when private agents maximize profits from cattle production at each site. The dynamical system is:

$$P_t = \left[gWP^{\eta}-d\left(1+\delta P\right)-A\right]P+D_PP_{zz}=f\left(P,W\right)+D_PP_{zz} \tag{60}$$
$$W_t = \left(\beta+\zeta P\right)R-uWP-r_WW+D_WW_{zz}=g\left(P,W\right)+D_WW_{zz} \tag{61}$$

It should be noted that the system (60), (61) can also describe spatiotemporal evolution at the 'undisturbed Nature' case, if A is replaced by c_{avg}. Thus differences between the 'undisturbed Nature' and the the two types of POMP (enforceable property rights or open access) can be analyzed in

[20]These simplifying assumptions are useful for providing a structure that helps to develop simulations which support our theoretical findings, without affecting the main characteristics of the problem.

[21]Alternatively we could substitute open access harvesting $\hat{H}$. The analysis would be the same.

terms of differences between natural parameters such as c_{avg}, and economic parameters such as (p, c, a).

To explore the formation of spatial patterns we apply the theory developed in section 2. The FSS $\left(P^0, W^0\right)$ is defined by the solution of the system

$$W = \frac{A + d\left(1 + \delta P\right)}{gP^{\eta}} \ , \ P = \frac{r_W W - \beta R}{\zeta P u W} \tag{62}$$

Thus $\left(P^0, W^0\right)$ depends on the economic variables through A.

Using the Jacobian of the flat system (where $D_P = D_W = 0$) evaluated at the FSS, the stability of this FSS requires

Trace conditions:
$\eta g W^0 \left(P^0\right)^{\eta-1} - d\delta > 0$[22]
$\mathrm{tr} J^P = \left(\eta g W^0 \left(P^0\right)^{\eta-1} - d\delta\right) P - \left(uP^0 + r_W\right) < 0$
Determinant conditions:
$\zeta R - uW^0 < 0$
$\det J_F = -\left(\eta g W^0 \left(P^0\right)^{\eta-1} - d\delta\right)\left(uP^0 + r_W\right) - \left(\zeta R - uW^0\right) > 0$
$J_{PF} = \begin{pmatrix} f_P & f_W \\ g_W & g_W \end{pmatrix} = \begin{pmatrix} \left(\eta g W^0 \left(P^0\right)^{\eta-1} - d\delta\right) P & gP^{\eta} \\ \left(\zeta R - uW^0\right) & -\left(uP^0 + r_W\right) \end{pmatrix}$

If local stability conditions are satisfied, then both eigenvalues of J_F are positive.

Pattern formation through the Turing mechanism requires, according to the conditions of theorem 1:

$$\eta g W^0 \left(P^0\right)^{\eta-1} - d\delta > \frac{D_P}{D_W}\left(uP^0 + r_W\right) \tag{63}$$

$$-\frac{\left[-\left(uP^0 + r_W\right) D_P + \left(\eta g W^0 \left(P^0\right)^{\eta-1} - d\delta\right) D_W\right]^2}{4 D_P D_W} + \det J^P < 0 \tag{64}$$

The last condition (64) is equivalent to having the dispersion relationship (16) being negative for a certain range of wavenumbers k. For these wavenumbers one eigenvalue of the Jacobian matrix[23]

$$J_{PS} = \begin{pmatrix} f_P - k^2 D_P & f_W \\ g_W & g_W - k^2 D_W \end{pmatrix}$$

becomes positive. Thus the FSS is unstable to spatial perturbations and a spatial pattern is formed around the FSS.

It should be noted that the emergence or not of patterns depends, apart from the parameters of the natural system, on economic variables which are summarized by A. For example, it is clear from (62) that a change in A will affect the FSS values $\left(P^0, W^0\right)$ and the pattern formation conditions (63), (64). Thus changes in economic parameters such as cost per unit of grazing effort, elasticities of the production function for cattle products, or the market price of cattle products, might create or break spatial patterns in the semi-arid grazing system.

Pattern formation can be explored in more detail by using numerical values for the economic parameters and the parameters of the natural system. For the natural system we use the parameter values:[24] $g = 0.001, \eta = 0.5, d = 0.03, \delta = 0.005, D_w = 1, \beta = 0.9, \zeta = 0.001, R = 50, u = 0.01, r_w = 0.1, D_p = 0.02$. For the economic parameter A we assume $c = 1, p = 1.1, a = 0.3$, which implies $A = 0.543$. The reaction-diffusion system has two flat steady states under enforceable property rights and profit maximization, which are shown in figure 2.

The steady states are $S_1^0 = \left(P_1^0, W_1^0\right) = (44.053, 87.327)$ and $S_2^0 = \left(P_2^0, W_2^0\right) = (2.53781, 359.927)$. Calculations of the eigenvalues for the linearization matrix for each FSS reveal that S_1^0 is locally stable, while S_2^0 has the saddle point property.[25] The vector field in the neghborhood of S_1^0 is shown in figure 3.

When spatially heterogeneous perturbations are introduced, the corresponding dispersion relation is: $\phi\left(k^2\right) = 0.02k^4 - 0.272k^2 + 0.0876$. As shown in figure 4 the dispersion relation becomes negative for positive values of the wave number k and thus the FSS is destabilized by spatial perturbations. Turing diffusion induced instability emerges under profit maximization and a spatial pattern is formed in the semi-arid system with the characteristics

[23] See also Appendix 1, (78).

[24] These values are in line with those used by van de Koppel et al. (2002).

[25] The eigenvalues are: $S_1^0 \to (-0.128667 \pm 0.266616i)$, $S_2^0 \to (0.247865, -0.0869336)$.

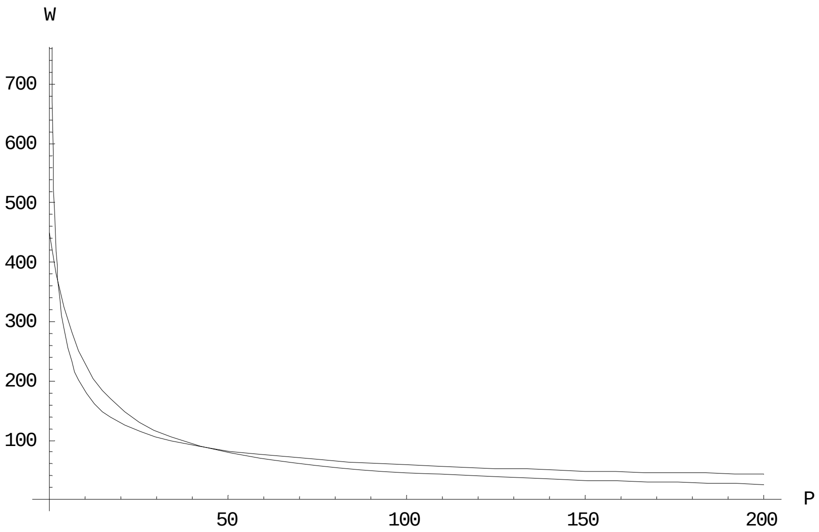

Figure 2: Flat Steady States at the POMP

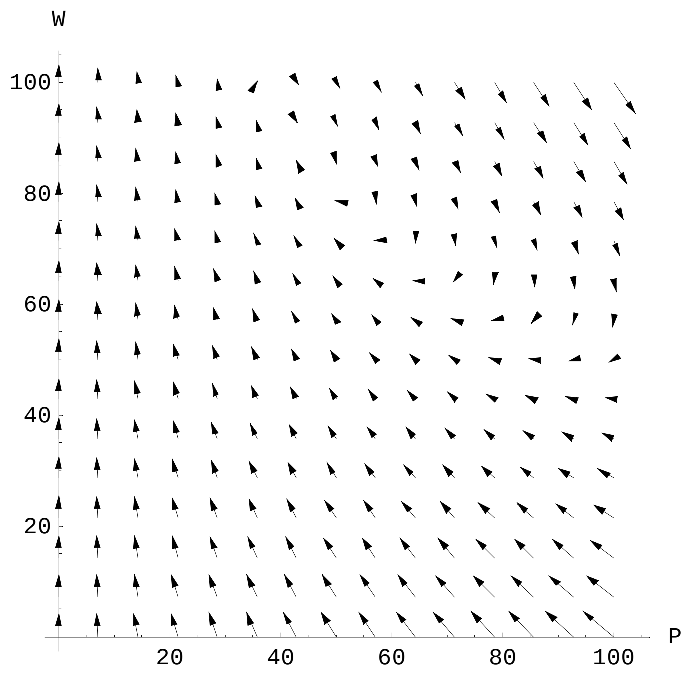

Figure 3: Vector field around the stable FSS

of our numerical example.

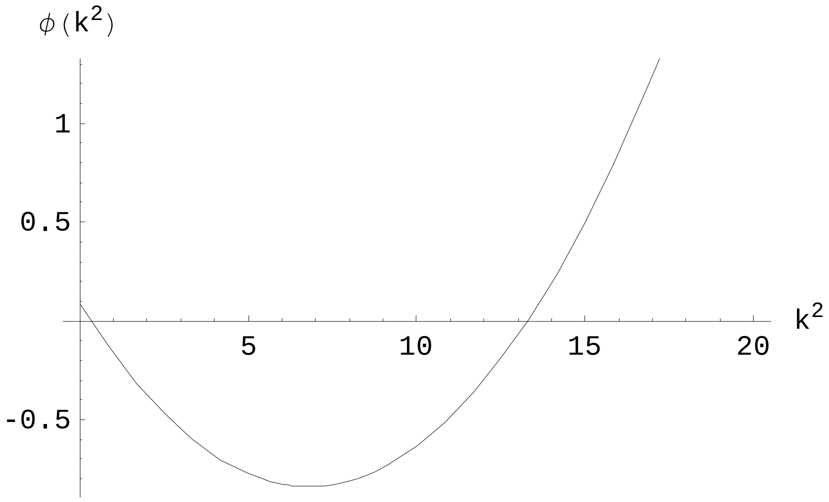

Figure 4: The dispersion relationship

In the dispersion relationship of figure 4 $k^2_{\min} = 6.81$. The eigenvalue $\sigma\left(k^2_{\min}\right) = 0.114757$ is the eigenvalue which determines the fastest temporal growth of the spatial perturbation. Thus the spatial perturbation does not die out as time passes but grows to form a spatial pattern. To obtain a picture of the growing spatial instability on the neighborhood of the FSS we assume that the size of the spatial domain is $L = 3$ so that the range of unstable wave numbers admits only the wave number $n = 1$.[26] Then the growing spatial instability for the plant biomass and soil water is proportional to $\exp\sigma\left(k^2_{\min}\right)\cos\left(2\pi z/3\right)$ and is given approximately by

$$\begin{aligned} P\left(t,z\right) &\sim 44.053 + B_p v^1 \exp(0.114757t)\cos\left(\frac{2\pi z}{3}\right) \ , \ v^1 = 0.99 \\ P\left(t,z\right) &\sim 87.327 + B_w v^2 \exp(0.114757t)\cos\left(\frac{2\pi z}{3}\right) \ , \ v^2 = -0.11 \end{aligned}$$

where $v^j, j = 1,2$ is the first and the second component of the eigenvector which corresponds to the eigenvalue $\sigma\left(k^2_{\min}\right) = 0.114757$ and $B's$ are deter-

[26] The size of the spatial domain satisfies in this case $n_1 \geq (Lk_1/2\pi)$ and $n_2 \leq (Lk_2/2\pi)$, where $k_1 = \sqrt{0.3297}, k_2 = \sqrt{13.2895}$. k_1^2 and k_2^2 are the zeroes of the dispersion relationship in figure 4.

mined by initial conditions. The approximate evolution of the two growing instabilities are shown in figures 5 and 6.

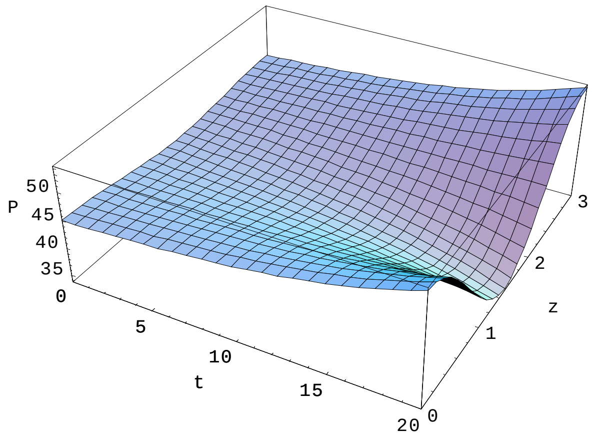

Figure 5: Spatial instability:Plant biomass

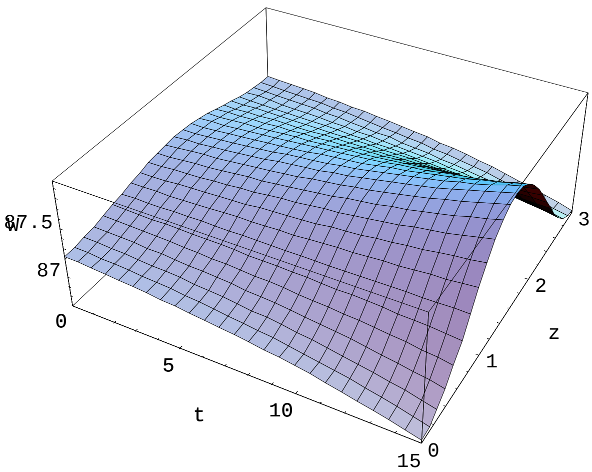

Figure 6: Spatial instability: Soil water

To obtain an idea about the impact of the economic parameters on pattern formation we show dispersion relations corresponding to different values of A in figure 7.

The three curves correspond to $A = 0.6, 0.5$ and 0.32. As A is reduced the dispersion relationship moves upwards. For $A = 0.32$, $k^2_{\min} > 0$ and the FSS, which for this case turns out to be a stable node, is not destabilized by spatial

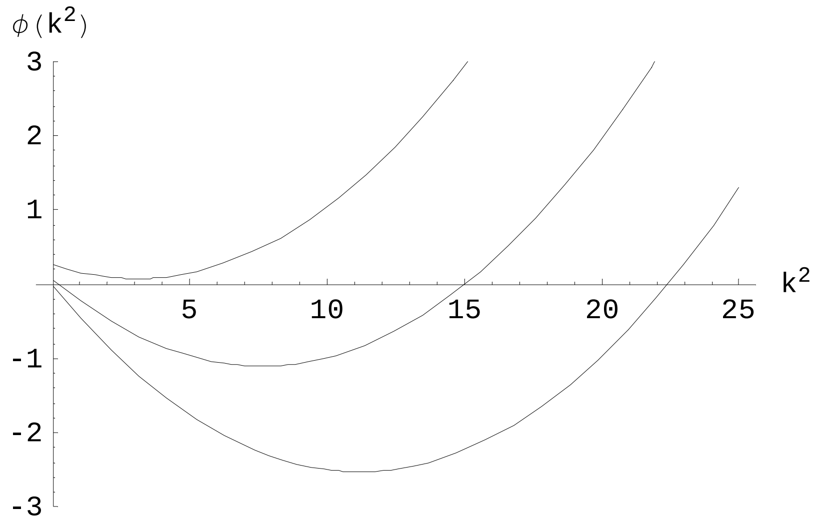

Figure 7: Dispersion relationships and changes in harvesting

perturbations. Thus a spatial pattern does not emerge and plant vegetation is *spatially homogeneous* in the given space domain. This result leads us, in the context of the specific model used, to the following proposition:

Proposition 1 *Let harvesting at all sites of the semi-arid system be proportional to plant biomass at the site, or* $H(t,z) = \Phi P(t,z)$, *where* Φ *is determined (i) by herbivore abundance and specific consumption in undisturbed Nature or (ii) by economic parameters in profit maximizing equilibrium with full property rights, or in an open access equilibrium. The higher* Φ *is, the more likely that spatial patterns will emerge.*

The intuition behind this result can be described in the following way. When humans are not in the picture Nature designs, under conditions of small and most likely erratic rainfall, heterogeneous patterns of vegetation to maximize water capture for promotion of plant growth on little "islands". Thus when there is not enough water patchiness is an essential design by Nature to capture what little rainfall there is to support, at least some patches of vegetation since without such patchiness design there would be no vegetation at all. We can view Nature as "optimizing" total plant production over the set of "designs" of the landscape. In this case "designs" are moving water around at a transport cost as well as moving nutrients around over "patches" at transport costs of nutrients under evaporation, leaching, and other losses. Because thresholds have to be crossed by soil water and other soil nutrients to get any growth at all on a patch, Nature "designs" heterogeneity patterns to "optimize" plant production.

If we add now 'graziers' private agents which harvest plant biomass„ these graziers can be treated like "evaporators" on a patch and also a bit like "nutrient/soil water" removers too (hoof compactification of the soil causes it to infill water more slowly, etc.). Assume, to simplify things in the analysis of the graziers impacts, that we have a discrete two patch model and assume that Nature optimizes plant production in the sense of the paragraph above by having vegetation only on one patch. When grazing is introduced a grazier on patch 1 may impede Nature's transport of water to patch 2 so both patches die, not just one. If we allow for a continuum of patches as we do in our model, then heterogeneity patterns will emerge with different concentration of plant biomass in different sites. It is most likely that patches of vegetation smaller in size and less in number emerge when graziers are introduced. Thus under grazing, the spatial pattern of vegetation is expected to be coarser relative to pattern emerging in the state of undisturbed Nature. It could be the case, in a seasonal model with wet and dry seasons, that without graziers vegetation is spatially homogenous in the wet season, since there is plenty of water to support vegetation in all patches, but 'patchiness' emerges in the wet season when graziers are introduced. In our model this process is captured by the downward movements of the dispersion relationship as the harvesting parameter Φ increases under the influence of graziers. Under open access grazing pressure is even higher, relative to profit maximization under enforceable property rights, which means that probably more patches will die out and the pattern of vegetation will be even coarser. Thus forces inducing spatial patterns and patchiness in the semi-arid system are expected to be the strongest under open access conditions.

Following the argument of the above paragraph the impact of changes in rainfall, which is an important driver of the system, on pattern formation can be traced through changes in R. Changes in rainfall in semi-arid systems introduce seasonality and our model can be used to study pattern formation in 'dry' or 'wet' seasons by studying solutions under parametric changes in R. Figure 8 shows three dispersion relationships resulting from setting $R = 40, 50, 100$ keeping the harvesting parameter at $A = 0.543$.

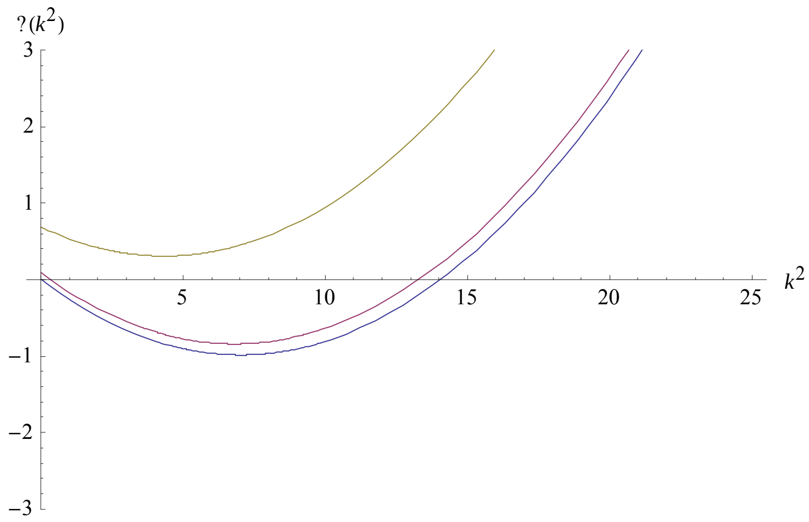

Figure 8: Dispersion relationships and changes in rainfall

The dispersion relation shifts upward as rainfall increases. Thus a 'wet' period with $R = 100$ will result in spatially homogeneous plant vegetation.[27] Combined with Proposition 1, this result suggests that aggressive harvesting could result in spatially heterogeneous plant vegetation even in 'wet' systems.

Having established that a spatial pattern is emerging in the neighborhood of the FSS, due to the spatial perturbation, we seek to explore the structure of a spatially heterogeneous steady state. As noticed in section 2, although the state variables grow exponentially in the neighborhood of the FSS due to the spatial perturbation, the kinetics of the system (60), (61) could bound the solution $(P(t,z), W(t,z))$ which eventually would settle to a spatial pattern. Thus we need to examine whether in our numerical example the kinetics of the system (60), (61), bound the solution $(P(t,z), W(t,z))$ as t grows. Figures 9 and 10 depict the solution $(P(t,z), W(t,z))$ obtained by numerically solving the system of partial differential equations corresponding to (60), (61).[28]

The behavior of the solution confirms the hypothesis that there is tem-

[27] This result is in line with results obtained by Vishwesha Guttal and Jayaprakash (2007).

[28] We use as boundary conditions, $P(0,z) = 44.053 + \cos(2\pi z/3)$, $W(0,z) = 87.3266 + \cos(2\pi z/3)$ which correspond to the growing spatial instability of figures 5 and 6 for $t = 0$ with $L = 3$, and the circle boundary conditions $P(t,0) = P(t,3)$, $W(t,0) = W(t,3)$.

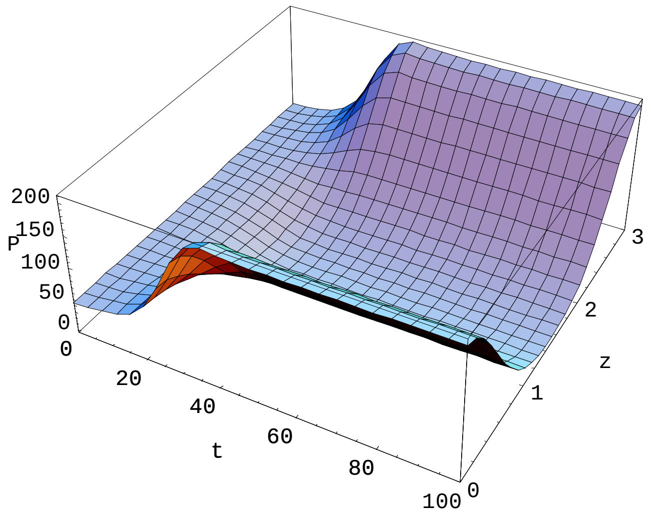

Figure 9: Long-term spatiotemporal evolution of plant biomass

poral growth near the FSS but then the kinetics confine the solution which settle to a clearly spatially heterogeneous steady state, with $P(100,0) = P(100,3) = 210.523$, $P(100,1.5) = 5.54$, $W(100,0) = W(100,3) = 48.489$, $W(100,1.5) = 67.344$. Thus there is high concentration of plant biomass in a one point and low concentration in the middle of the circle. Water concentration follows the opposite pattern. The spatially heterogeneous steady state can also be obtained by solving the second-order system (60), (61) in the spatial variable z for $P_t = W_t = 0$, or

$$\frac{d^2P(z)}{dz^2} = -\frac{1}{D_P}\left[gWP^{\eta} - d(1+\delta P) - A\right]P \tag{65}$$

$$\frac{d^2W(z)}{dz^2} = -\frac{1}{D_W}\left[(\beta + \zeta P)R - uWP - r_W W\right] \tag{66}$$

with initial spatial conditions $(P(0), W(0)) = (210.523, 48.489)$.[29] The solution is shown in figure 11.

The plant biomass follows the U pattern, while the water stock follows the opposite pattern.

[29] We use arbitrary initial conditions for $P'(0)$, $W'(0)$ and multiple shooting in order to obtain the solution which satisfies the circle conditions $P(0) = P(3)$, $W(0) = W(3)$.

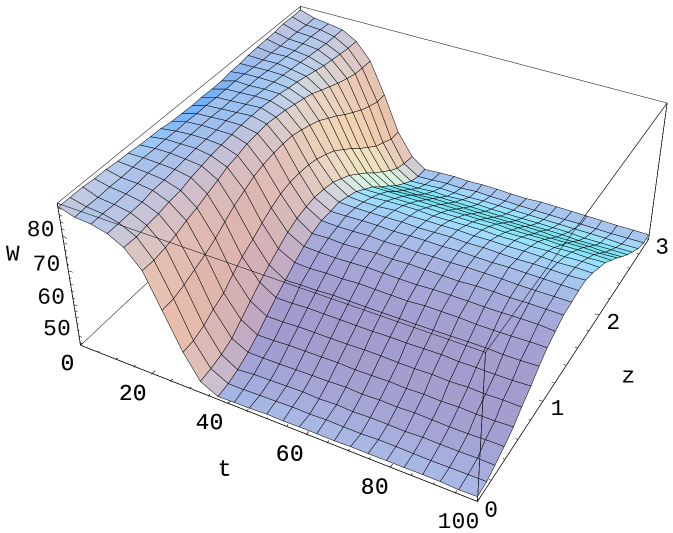

Figure 10: Long-term spatiotemporal evolution of soil water

5.2 Pattern Formation in a Semi-arid System under Social Objectives

We consider now the management of a semi-arid system under a social planner that takes into account both the dynamics of the system and the spatial externality. The Hamiltonian for the SOMP is

$$\begin{aligned}\mathcal{H} &= pP^{\alpha}E^{1-\alpha} - cE + \lambda\left[\left(gWP^{\eta} - d\left(1+\delta P\right)\right)P - TH + D_P P_{zz}\right] \\ &\quad + \mu\left[\left(\beta + \zeta P\right)R - uWP - r_W W + D_W W_{zz}\right]\end{aligned}$$

and the maximum principle implies:

$$(1-\alpha)\,pP^{\alpha}E^{-\alpha} - c - \lambda = 0 \Rightarrow E^{s}(t,z) = \gamma^{S}P(t,z) \tag{67}$$

$$\gamma^{S} = \left[\frac{c}{(p-\lambda)(1-a)}\right]^{-\frac{1}{a}},\ TH^{S}(t,z) = A^{*}P(t,z),\ A^{*} = \left(\gamma^{S}\right)^{1-\alpha} \tag{68}$$

$$P_t = \left(gWP^{\eta} - d\left(1+\delta P\right) - A^{*}\right)P + D_P P_{zz} \tag{69}$$

$$W_t = \left(\beta + \zeta P\right)R - uWP - r_W W + D_W W_{zz} \tag{70}$$

$$\begin{aligned}\lambda_t &= \left[\rho - (1+\eta)gWP^{\eta} + d\left(1 - 2\delta P\right) + A^{*}\right]\lambda - \\ &\quad \mu\left(\zeta R - uW\right) - \alpha pP^{\alpha-1}\left(E^{S}\right)^{1-\alpha} - D_P P_{zz}\end{aligned} \tag{71}$$

$$\mu_t = \left[\rho + \left(uP + r_W\right)\right]\mu - \lambda gP^{\eta+1} - D_W W_{zz} \tag{72}$$

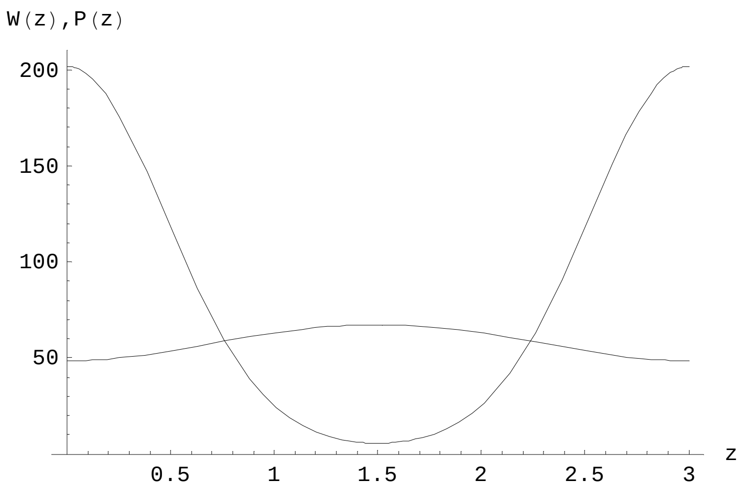

Figure 11: The Spatially Heterogeneous Steady State

Assume that a FOSS $(P^*, W^*, \lambda^*, \mu^*) : (P_t = W_t = \lambda_t = \mu_t = 0; D_P = D_W = 0)$ exists and that it has the saddle point property with a two-dimensional stable manifold. Using (69) - (72), the linearization of the MHS around the FOSS under heterogeneous spatial perturbations of the form defined in (34) is

$$\begin{aligned}
\bar{\mathbf{q}}_t &= J^S \cdot \bar{\mathbf{q}}\ , J^S = \begin{pmatrix} H_{\mathbf{px}} - Dk^2 & H_{\mathbf{pp}} \\ -H_{\mathbf{xx}} & \rho I_2 - H_{\mathbf{xp}} + Dk^2 \end{pmatrix}, D = \begin{pmatrix} D_p & 0 \\ 0 & D_w \end{pmatrix} \\
\bar{\mathbf{q}}_t &= \left(\bar{P}_t, \bar{W}_t, \bar{\lambda}_t, \bar{\mu}_t\right)', \mathbf{q} = (P, W, \lambda, \mu), \mathbf{x} = (P, W), \mathbf{p} = (\lambda, \mu)
\end{aligned}$$

where H denotes the 'flat' Hamiltonian obtained by setting $D_P = D_W = 0$. Pattern formation through optimal diffusion induced instability is determined by the conditions of theorem 2. To obtain more insights into the structure of the solution for the SOMP we continue the numerical simulation of the previous section. With discount rate $\rho = 0.03$ and the same parameter values resulting in the FSS $\left(P_1^0, W_1^0\right) = (44.053, 87.327)$ for the POMP, the FOSS for the SOMP is: $(P^*, W^*, \lambda^*, \mu^*) = (956.24, 9.61, 0.517, 1.577)$ with harvesting parameter $A^* = 0.124$ The FOSS is a saddle point with real eigenvalues $\boldsymbol{\sigma} = (9.547, -9.517, 0.294, -0.264)$. The SOMP accumulates more biomass and uses more water relative to the POMP, while at the steady state 12.4% of this biomass is harvested relative to 53.4% at the POMP. This is the expected result when the dynamics are taken into account. To examine for the emergence of pattern formation the quantities which are central are the dispersion relationships $K\left(k^2\right)$ and $J^S\left(k^2\right)$ and the constant K^0, which are defined as:

$$K^0 = \left[\left(\rho - H_{P\lambda}\right) H_{\lambda P} + H_{PP} H_{\lambda\lambda}\right] + \tag{73}$$

$$\left[\left(\rho - H_{W\mu}\right) H_{\mu W} + H_{WW} H_{\mu\mu}\right] - H_{\lambda W} H_{P\mu} \tag{74}$$

$$K\left(k^2\right) = -\left(D_P^2 + D_W^2\right) k^4 + \tag{75}$$

$$\left[D_P\left(2H_{P\lambda} - \rho\right) + D_W\left(2H_{W\mu} - \rho\right)\right] k^2 + K^0 \tag{76}$$

$$J^S\left(k^2\right) = \begin{vmatrix} H_{\mathbf{px}} - Dk^2 & H_{\mathbf{pp}} \\ -H_{\mathbf{xx}} & \rho I_2 - H_{\mathbf{xp}} + Dk^2 \end{vmatrix} \tag{77}$$

It turns out that for the parameter values used before, and also for a set of parameter values in the neighborhood of the original set, the conditions of theorem 2 are not satisfied. $K^0 < 0$ by the saddle point property of the FOSS and for the relevant parameter set $D_P\left(2H_{P\lambda} - \rho\right) + D_W\left(2H_{W\mu} - \rho\right) < 0$. Thus the dispersion relationship (75) attains the maximum for $k^2 < 0$ which means that the conditions of part (a) of theorem 2 are not satisfied. For the dispersion relationship (77), $J^S(0) > 0$ by the saddle point property of the FOSS. $J^S\left(k^2\right)$ is monotonically increasing in k^2 for the relevant parameter set. Thus the conditions of part (b) of theorem 2 are not satisfied. Similarly the conditions of part (c) of theorem 2 are not satisfied for the relevant parameter set.

This result suggests that, for our example, optimal management by taking into account both the dynamics and the spatial externality tends to produce a flat system and break the patterns in plant biomass created under private optimization objectives. Thus the POMP generates patchiness in vegetation while the SOMP with the same water resources, but less aggressive harvesting relative to the POMP, generates homogenous vegetation on the spatial domain. Linear approximation of the optimal time paths for the state and costate variables under the assumption that the initial state for plant biomass and ground water coincides with the FSS under private optimization are

$P(t) = 956.24 - 234.063 \exp(-9.517t) - 689.124 \exp(-0.264t), \forall z.$

$W(t) = 9.61 + 74.339 \exp(-9.517t) + 3.377 \exp(-0.264t), \forall z.$

$\lambda(t) = 0.517 + 0.0648 \exp(-9.517t) + 0.1822 \exp(-0.264t), \forall z.$

$\mu(t) = 1.577 + 0.00027 \exp(-9.517t) + 0.2671 \exp(-0.264t), \forall z.$

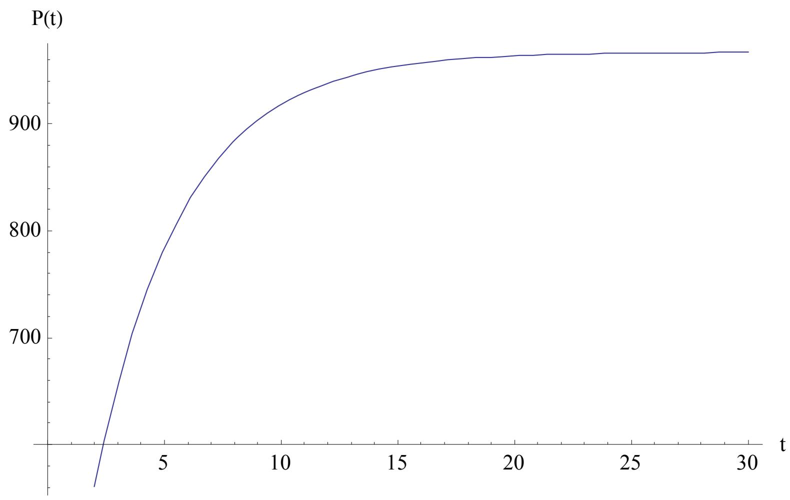

Figure 12: Optimal path for the plant biomass

The time paths are shown in figures 12, 13, and 14 .

To examine conditions under which a spatial pattern might emerge at the SOMP we consider the case of an extremely dry system by setting $R = 5$. The FOSS is $(P^*, W^*, \lambda^*, \mu^*) = (20.907, 14.898, 0.76049, 0.21441)$ with harvesting parameter $A^* = 0.035$. This FOSS is a saddle point with real eigenvalues $\boldsymbol{\sigma} = (0.256528, -0.226528, 0.120913, -0.0909126)$, so the flat system can be controlled to converge at the FOSS.

When a spatial perturbation characterized by high soil water diffusion $(D_w = 10, D_P = 0.05)$ is introduced into the dry system, the FOSS is destabilized in the sense of part (b) of theorem 2. The dispersion relationship is shown in figure 15.

For $k^2 \in (0.0213, 2.118)$, $J^S\left(k^2\right) < 0$. Using $k^2 = 1$, the eigenvalues of the linearization matrix of the system under the spatial perturbation is $\boldsymbol{\sigma} = (10.3377, -10.3077, 0.015 \pm 0.0807i)$, which implies destabilization according to part (b) of theorem 2. Using the eigenspace of the two real eigenvalues and setting $L = 7$ so that only one wavenumber exists, the spatiotemporal paths for P and W in the neighborhood of the FOSS can be

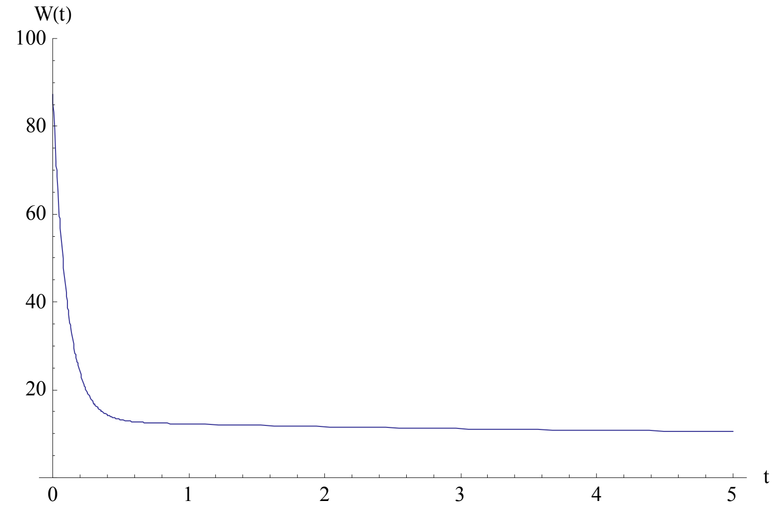

Figure 13: Optimal path for soil water

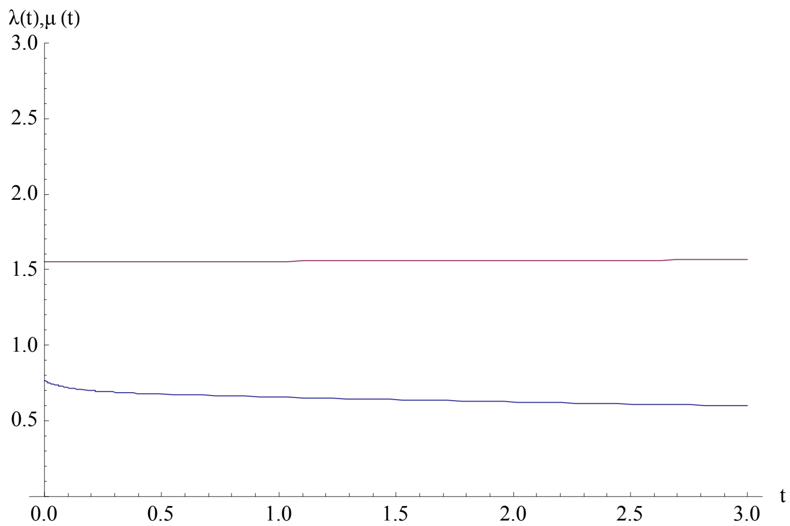

Figure 14: Time paths for biomass and soil water shadow prices

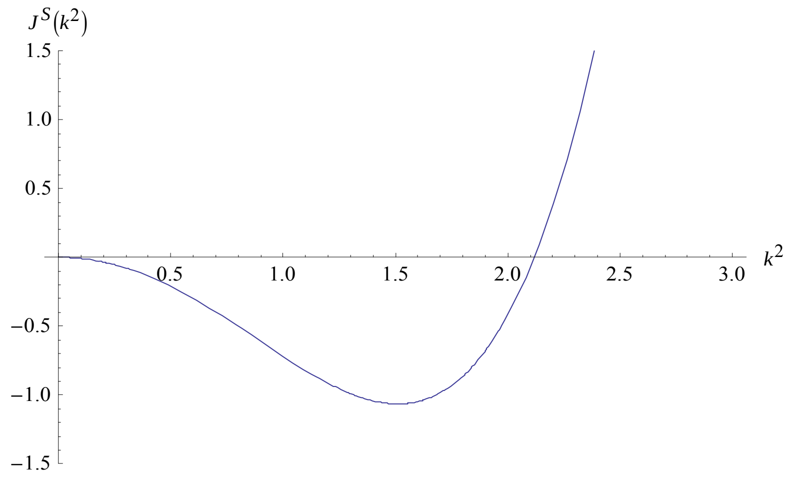

Figure 15: Dispersion relationship at the SOMP for a dry system.

approximated as:

$$
\begin{aligned}
P(t,z) &\sim 20.907 - 0.00679 \exp(10.3377t) \cos(2\pi z/7) \\
&\quad +0.00943 \exp(-10.3077t) \cos(2\pi z/7) \\
W(t,z) &\sim 14.898 + 0.000047 \exp(10.3377t) \cos(2\pi z/7) \\
&\quad -0.99995 \exp(-10.3077t) \cos(2\pi z/7)
\end{aligned}
$$

The spatiotemporal paths in the neighborhood of the FOSS are shown in figures 16 and 17.

To examine the long-run behavior of this extremely 'dry' system under spatial diffusion we explore the numerical solution of the MHS which consists of the four partial differential equations (69) - (72).[30] The results indicate that the system is not converging to a HOSS but instead it breaks down in a short period of time. It seems that the combination of spatiotemporal fluctuations induced by the high diffusion rates and the extremely dry conditions make the system collapse. This breakdown is approximatelly depicted in figures 18 and 19.

[30] We use as boundary conditions, the equations for the growing spatial instability in the neighborhood of the FOSS for $t = 0$, with $L = 7$, and the circle boundary conditions.

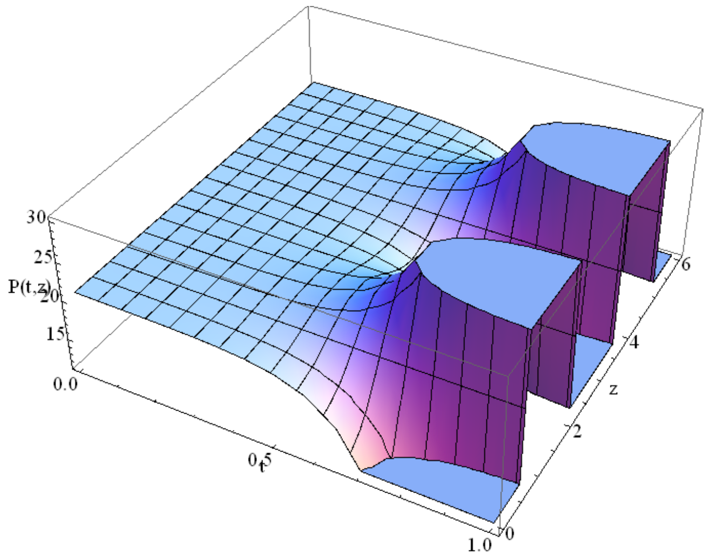

Figure 16: Spatial instability at the SOMP: Plant biomass

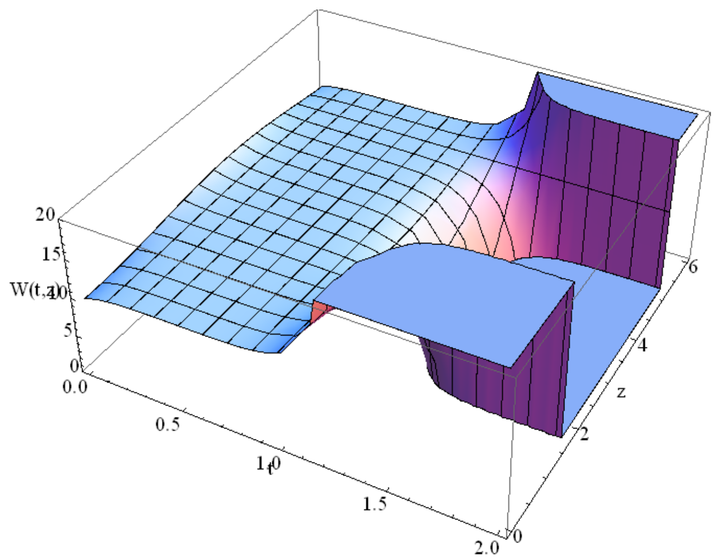

Figure 17: Spatial instability at the SOMP: Soil water

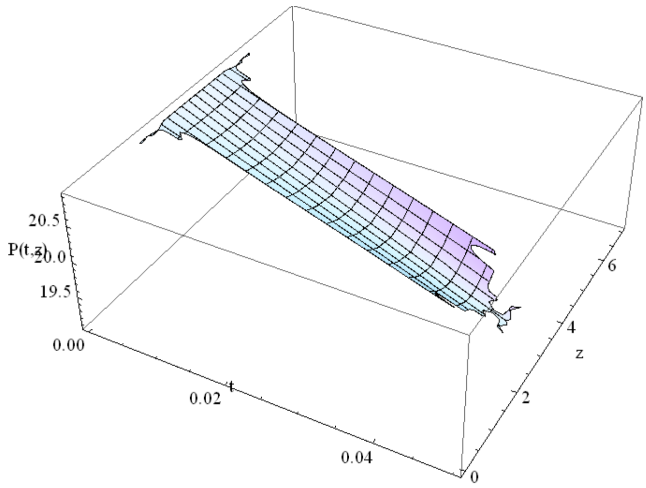

Figure 18: Collapse of plant biomass

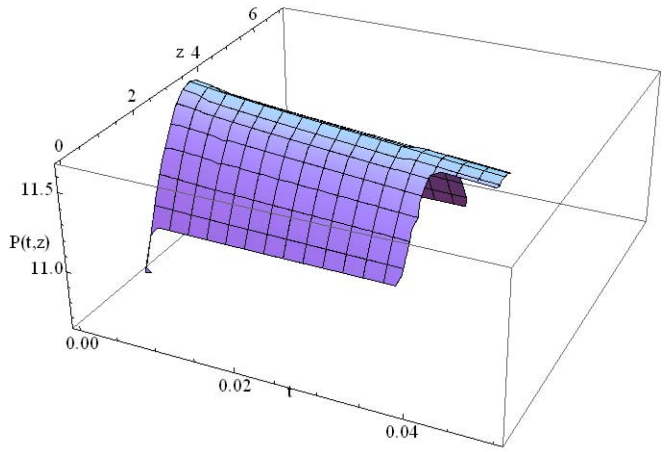

Figure 19: Collapse of soil water

5.3 Optimal Regulation of a Semi-Arid System

The state where water inflow is set at the intermediate value of $R = 50$, and the strength of spatial diffusion is such that a steady state spatially heterogeneous pattern occurs under private objectives, while a spatially homogenous system emerges under social objective with a much higher plant biomass level, supports the idea of regulating the system under private objectives so that the socially optimal steady state is attained. As discussed in section 4 regulation could take a quantity form, which implies harvesting limits, or a price form, which implies a tax on grazing effort.

To study the quantity instrument first, we recall that under profit optimization and social optimization the harvesting rules are

$$\begin{aligned} TH^0(t,z) &= AP(t,z), \ A = \left(\frac{c}{p(1-a)}\right)^{-\frac{1-a}{a}}, A^0 = 0.543 \\ TH^*(t,z) &= A^*P(t,z), \ A^* = \left[\frac{c}{(1-\alpha)(p-\lambda)}\right]^{-\frac{1-a}{a}}, A^* = 0.124 \end{aligned}$$

where the numerical values correspond to the specific steady harvesting rule in our numerical example.

Definition 1 *We define the **optimal harvesting limit** to be a value A^* such that the steady state of the plant biomass-soil water system (60), (61), with private agents maximizing harvesting profits under this limit using the rule $H^0(t,z) = A^*P(t,z)$, is the same as the socially optimal plant biomass-soil water steady state which is obtained from the MHS (69) - (72).*

Thus for the exact value of A^*, which is $A^* = 0.123595$ the regulated systems attainsc the flat socially optimal steady state (FSOSS) $(P^*, W^*) = (956.24, 9.61)$. Thus the regulated system tends to a homogeneous steady state. This is confirmed by the numerical solution of the system of partial differential equations (60), (61) for $A^* = 123595$. The results are shown in figures 20 and 21.

Any initial spatial perturbation 'dies out' at the regulated system, which converges to the FSOSS. By comparing these figures with figures 9 and 10 which depict the unregulated system, the impact of optimal regulation becomes clear in terms os the spatiotemporal evolution of the state variables becomes clear. The harvesting limit in quantity terms, that is the allowed amount of harvested plant biomass per unit area implied by rule A^*, could be

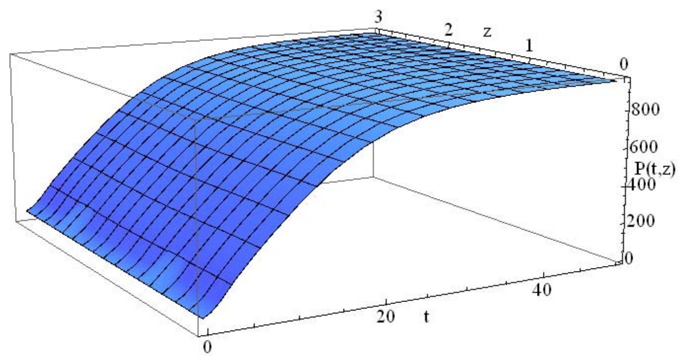

Figure 20: Plant biomass under optimal quantity regulation

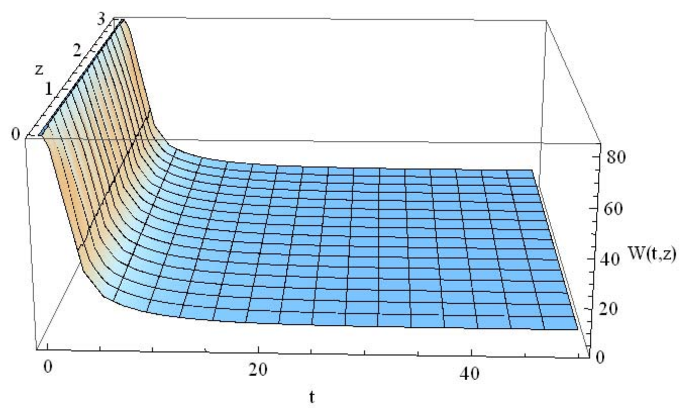

Figure 21: Soil water under optimal quantity regulation

implemented either by a command and control approach where the allowed harvesting at each site does not exceed the allowed harvest limit, or by setting up markets for tradable harvesting quotas where the total amount of quotas allowed to be traded at each site will not exceed the allowed harvest limit. Since the optimal regulation implies a spatially homogeneous state, limits will be the same across sites.

We turn now to price regulation which takes the form of a tax τ^* on grazing effort. Under this tax private agents solve the problem

$$\max_{\tau^*} p\left[P\left(t,z\right)\right]^{\alpha}\left[E\left(t,z\right)\right]^{1-\alpha} - cE\left(t,z\right) - \tau^* E\left(t,z\right)$$

leading to a harvesting $H^0_{\tau^*} = A_{\tau^*} P\left(t,z\right)$.

Definition 2 *We define the* ***optimal tax on grazing effort*** *to be a linear tax τ^*such that the steady state of the plant biomass-soil water system (60), (61), with private agents maximizing harvesting profits under this tax, is the same as the socially optimal plant biomass-soil water steady state which is obtained from the MHS (69) - (72).*

To obtain the optimal tax we use the fact that $A^* = \left[\frac{c}{(1-\alpha)(p-\lambda)}\right]^{-(1-a)/a}$. Then the optimal tax is set at the level τ^* such that $\left[\frac{c+\tau^*}{(1-\alpha)p}\right]^{-(1-a)/a} = A^*$. It is clear by its construction that the regulated, through the optimal tax, plant biomass-soil water system will attain the socially optimal steady state in the way shown in figures 20 and 21. In our numerical example the optimal tax is $\tau^* = 1.97214$.

The intuition behind the type of regulation described above can be clarified using again the two-patches example. Assume that under open access grazing only one patch can support vegetation, since there is not enough water to support vegetation in both patches. If however grazing is reduced in both patches, then there is the possibility that both patches will support vegetation. So the regulator has to design either quantity or price regulation in order to restrict grazing in both patches so that vegetation can be supported in both of them. In the optimal version of the regulation, grazing has to be reduced up to the point where vegetation in both patches will coincide with the steady state attained under social optimization.

6 Concluding Remarks

In this paper we develop a novel theoretical framework for studying ecosystems managed by humans in which state variables diffuse in space and interact among themselves. Arid and semi-arid grazing systems is a classic examples of such systems. We develop conditions for pattern formation in systems where economic agents maximize profit in each site without taking into account the impact of their actions on other sites, as well as conditions for pattern formation under social optimization objectives. We identify a diffusion induced spatial externality associated with a POMP, and two different mechanisms for pattern formation: the classic Turing mechanism applying to the POMP and optimal diffusion instability applying to the SOMP. These two mechanisms combined with spatiotemporal externalities create a divergence in the emerging solutions between the POMP and the SOMP, and a need for regulation. Using the solution of the SOMP we develop optimal spatiotemporal regulation which internalizes the externalities. We apply our theory to the management of a semi-arid system. Numerical simulations confirm our theoretical findings.

Appendix 1

After substituting (12) into (11) the linearization becomes

$$\bar{\mathbf{x}}_t(t,z) = J^L \bar{\mathbf{x}}(t,z) \ , J^L = \begin{pmatrix} b_{11} - D_{x_1}k^2 & b_{12} \\ b_{21} & b_{22} - D_{x_2}k^2 \end{pmatrix} \tag{78}$$

Since $\mathrm{tr} J^L = b_{11} + b_{22} - D_{x_1}k^2 - D_{x_2}k^2 < 0$, destabilization of the FSS under spatially heterogenous perturbations requires that

$$\det J^L = \phi\left(k^2\right) = D_{x_1}D_{x_2}k^4 - \left(b_{11}D_{x_2} + b_{22}D_{x_1}\right)k^2 + \det J^P < 0\,, \det J^P > 0 \tag{79}$$

where $\det J^P > 0$ by the stability assumption about the FSS. Relationship (79) is a dispersion relationship. The instability requirement will be satisfied if there exist wavenumbers k_1 and k_2 such that $\phi\left(k^2\right) < 0$ for $k^2 \in \left(k_1^2, k_2^2\right)$, which implies that $\sigma\left(k^2\right) > 0$ for $k^2 \in \left(k_1^2, k_2^2\right)$. This in turn requires that: (i) $k_{\min}^2$ which corresponds to the wavenumber which maximizes $\phi\left(k^2\right)$ is positive and, (ii) $\phi\left(k_{\min}^2\right) < 0$ or

$$\frac{b_{22}D_{x_1} + b_{11}D_{x_2}}{2D_{x_1}D_{x_2}} > 0 \tag{80}$$

$$-\frac{\left(b_{22}D_{x_1} + b_{11}D_{x_2}\right)^2}{4D_{x_1}D_{x_2}} + \det J^P < 0 \tag{81}$$

■

Appendix 2

Substituting the spatially heterogenous perturbations (34) into (32) we obtain the system

$$\begin{pmatrix} \bar{\mathbf{x}}_t(t,z) \\ \bar{\mathbf{p}}_t(t,z) \end{pmatrix} = J^S \begin{pmatrix} \bar{\mathbf{x}}(t,z) \\ \bar{\mathbf{p}}(t,z) \end{pmatrix} \ , \ J^S = \begin{pmatrix} H_{\mathbf{px}} - Dk^2 I_2 & H_{\mathbf{pp}} \\ -H_{\mathbf{xx}} & \rho - H_{\mathbf{xp}} + Dk^2 I_2 \end{pmatrix}$$

Define the matrix

$$Z\left(\frac{\rho}{2}\right) = \begin{pmatrix} H_{\mathbf{px}} - Dk^2 I_2 - \frac{\rho}{2} I_2 & H_{\mathbf{pp}} \\ -H_{\mathbf{xx}} & -H_{\mathbf{xp}} + Dk^2 I_2 + \frac{\rho}{2} I_2 \end{pmatrix}$$

By applying Kurz (1968, theorem 2) we obtain that if σ_1, σ_2 are eigenvalues of J, then they satisfy $\sigma_{1,2} = \frac{\rho}{2} \pm \psi$, where ψ is a pair of eigenvalues for Z. The eigenvalues of matrix Z are determined by the solution of the characteristic equation (e.g. Samuelson 1947, p. 373)

$$\psi^4 - M_3\psi^3 + M_2\psi^2 - M_1\psi + \det Z = 0 \tag{82}$$

where $M_3 = \mathrm{tr}(Z) = 0$. By rather tedious calculation we can further obtain: $M_2 = \left(K - \frac{\rho^2}{2}\right)$, with K defined in (**??**), and with M_2 being the sum of six principal minors of Z of second order; $M_3 = 0$, with M_3 being the sum of four principal minors of Z of third order; and $\det Z = \left(\frac{\rho}{2}\right)^4 - \left(\frac{\rho}{2}\right)^2 K + \det J$. Substituting in (82) and using the Kurz theorem we obtain the eigenvalues of J as:

$$ {}_1^3\sigma_2^4 = \frac{\rho}{2} \pm \sqrt{\left(\frac{\rho}{2}\right)^2 - \frac{K}{2} \pm \sqrt{\left(\frac{K}{2}\right)^2 - \det J^S}} \tag{83}$$

which is an extension of Dockner's (1985) formula for the eigenvalues of the MHS for optimal control problems with two state variables, for the case where the state variables diffuse in space.

We can now prove part (a) of the theorem:

The FOSS will have the saddle point property (two positive and two negative eigenvalues) under spatially heterogenous perturbations if (i) $K < 0$ and (ii) $0 < \det J < \left(\frac{K}{2}\right)^2$ (Dockner 1985). In this case $\det J > 0$. Therefore, if $K > 0$ while (ii) is still satisfied, the two negative eigenvalues will become positive. From the definition of $K = K_1 + K_2 + 2K_3$ in (**??**) and the definition of K^0 in (37) we obtain

$$\begin{aligned} K_i\left(k^2\right) &= K_i^0 + \left[-D_{x_i}^2 k^4 + D_{x_i}k^2(2H_{x_ip_i} - \rho)\right] \ , i = 1,2 \ , K_3 = K_3^0 \\ K_i^0 &= \left(\rho - H_{x_ip_i}\right) H_{p_ix_i} + H_{x_ix_i}H_{p_ip_i} \ , K_3 = -H_{p_1x_2}H_{x_1p_2} + H_{x_1x_2}H_{p_1p_2} \\ K^0 &= \sum_{i=1,2} \left[\left(\rho - H_{x_ip_i}\right) H_{p_ix_i} + H_{x_ix_i}H_{p_ip_i}\right] + \left[-H_{p_1x_2}H_{x_1p_2} + H_{x_1x_2}H_{p_1p_2}\right] \end{aligned} \tag{84}$$

Therefore,

$$K\left(k^2\right) = -\left(D_{x_1}^2 + D_{x_2}^2\right) k^4 + \left[\sum_{i=1,2} D_{x_i}(2H_{x_ip_i} - \rho)\right] k^2 + K^0 \ , K^0 < 0 \tag{85}$$

where $K^0 < 0$ because of the saddle point assumption for the FOSS. For instability we want $K\left(k^2\right) > 0$ for some wavenumber k, thus (85) is a dispersion relationship. But:

$$K^{'}\left(k^2\right) = -2\left(D_{x_1}^2 + D_{x_2}^2\right)k^2 + \left[\sum_{i=1,2} D_{x_i}(2H_{x_i p_i} - \rho)\right]$$

and $K(0) = K^0 < 0$. Therefore $K^{'}(0) = \left[\sum_{i=1,2} D_{x_i}(2H_{x_i p_i} - \rho)\right]$ must be positive so that $K\left(k^2\right)$ cuts the y-axis at a negative point and is increasing. Furthermore, since $K\left(k^2\right)$ is strictly concave because $-\left(D_{x_1}^2 + D_{x_2}^2\right) < 0$ the maximum of $K\left(k^2\right)$ must be positive for instability.

The maximum of $K\left(k^2\right)$ is obtained at:

$$\begin{aligned} k_{\max}^2 &: \quad K^{'}\left(k_{\max}^2\right) = 0 \\ k_{\max}^2 &= \frac{\left[\sum_{i=1,2} D_{x_i}(2H_{x_i p_i} - \rho)\right]}{2\left(D_{x_1}^2 + D_{x_2}^2\right)} \end{aligned}$$

and for instability

$$K\left(k_{\max}^2\right) = -\left(D_{x_1}^2 + D_{x_2}^2\right)k_{\max}^4 + \left[\sum_{i=1,2} D_{x_i}(2H_{x_i p_i} - \rho)\right]k_{\max}^2 + K^0 > 0$$

or

$$-\left(D_{x_1}^2 + D_{x_2}^2\right)\frac{\left[\sum_{i=1,2} D_{x_i}(2H_{x_i p_i} - \rho)\right]^2}{4\left(D_{x_1}^2 + D_{x_2}^2\right)^2} + \frac{\left[\sum_{i=1,2} D_{x_i}(2H_{x_i p_i} - \rho)\right]^2}{2\left(D_{x_1}^2 + D_{x_2}^2\right)} + K^0 > 0$$

or

$$\frac{\left[\sum_{i=1,2} D_{x_i}(2H_{x_i p_i} - \rho)\right]^2}{4\left(D_{x_1}^2 + D_{x_2}^2\right)} + K^0 > 0$$

where $K^0 < 0$ by the saddle point assumption of the FOSS. Therefore, the two negative eigenvalues of the FOSS will turn positive under diffusion if:

$$\frac{\left[\sum_{i=1,2} D_{x_i}(2H_{x_i p_i}-\rho)\right]}{2\left(D_{x_1}^2+D_{x_2}^2\right)} > 0 \tag{86}$$

$$\frac{\left[\sum_{i=1,2} D_{x_i}(2H_{x_i p_i}-\rho)\right]^2}{4\left(D_{x_1}^2+D_{x_2}^2\right)}+K^0 > 0 \tag{87}$$

Parts (b) and (c) of the theorem follow from Feichtinger et al. (1994) where the conditions for obtaining three eigenvalues with positive real parts and four complex eigenvalues with positive real parts from (83) are stated.

Complex eigenvalues are determined as follows: If $\Phi = \left(\frac{K}{2}\right)^2 - \det J < 0$ then $\sqrt{\left(\frac{K}{2}\right)^2 - \det J} = i\sqrt{-\Phi} = i\Psi$. Write

$$\sqrt{\left(\frac{\rho}{2}\right)^2 - \frac{K}{2} \pm i\Psi} = \sqrt{\Theta \pm i\Psi}$$

Then by the definition of the square root of a complex number:

$$\sqrt{\Theta \pm i\Psi} = \sqrt{\frac{\zeta+\Theta}{2}} \pm i\frac{\Psi}{\sqrt{2\left(\zeta+\Theta\right)}}\ ,\zeta = \sqrt{\Theta^2+\Psi^2}$$

Then the eigenvalues of the linearization (32) are:

$${}_1^3\sigma_2^4 = \frac{\rho}{2} \pm \sqrt{\frac{\zeta+\Theta}{2}} \pm i\frac{\Psi}{\sqrt{2\left(\zeta+\Theta\right)}} \tag{88}$$

Four complex eigenvalues with positive real parts require that $\rho/2-\sqrt{\left(\zeta+\Theta\right)/2} > 0$ if $(\zeta+\Theta) > 0$. If $\zeta+\Theta < 0$ the second term of (88) will be imaginary but in any case all the eigenvalues are complex with positive real parts provided that the conditions (c) of theorem 2 are satisfied.■

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