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Optimal Management of Ecosystem Services with Pollution Traps: The Lake Model Revisited[1]

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Abstract

In this paper, optimal management of the lake model and common-property outcomes are reconsidered when the lake model is extended with a slowly changing variable (describing sedimentation and recycling of phosphorus), which is considered fixed in the simplified version that has been used in the literature up to now. Some intuitive results are reconfirmed but new optimal trajectories are found that were hidden in the simplified analysis. Moreover, it is shown that in the case of common-property, two Nash equilibria exist in a certain area of initial conditions but the one leading to the steady state with a high level of ecological services dominates the other one. However, for an adjacent area of initial conditions, only the Nash equilibrium steady state with a low level of ecological services exists. This implies that users of the lake are trapped in the bad Nash equilibrium as a result of initial conditions, leading to a substantial drop in welfare because the good Nash equilibrium is not reachable from the prevailing initial conditions. Finally, it is shown that the welfare losses of implementing the optimal phosphorus loadings from the simplified version into the full lake

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model are small, because of the fast-slow dynamics. The analytical tool that is used in this paper is an advanced extension of a Matlab solver for boundary value problems.

Key words: ecosystem services, lakes, multiple equilibria, pollution trap, optimal control, games, fast-slow dynamics

JEL codes: Q20, Q25, C61, C63, C73

1 Introduction

Resources that are embedded in ecological systems often have characteristics that complicate management. A famous example is the lake system that may have multiple equilibria for certain phosphorus loadings due to non-linear feedbacks in the eutrophication process in the lake (Carpenter and Cottingham, 1997, Scheffer, 1997). Gradually increasing the release of phosphorus on the lake may lead to a tipping point where the lake system flips from an oligotrophic state, with a high level of ecological services, to a eutrophic state, with a low level of services. These services include, for example, water, fish and several amenities. After a flip it may be costly (a hysteresis effect) or even impossible to restore the oligotrophic conditions. Many other systems such as coral reefs, grasslands and climate have similar characteristics.

The release of phosphorus on the lake is a decision variable and a dynamic trade-off has to be made between the agricultural benefits that are connected to the release of phosphorus and the loss of ecological services that are due to the accumulation of phosphorus in the water of the lake (Brock and Starrett, 2003, Mäler et al., 2003). Optimal management of this ecological system may have multiple steady states, that are comparable to the multiple equilibria for fixed loadings above. This may lead to history dependence in the sense that a so-called Skiba point or indifference point exists that divides the initial conditions of the lake into an area from where the optimal trajectory converges to an oligotrophic steady state and an area from where this trajectory converges to a eutrophic steady state.

Many other papers have been written on optimal management of the lake (e.g., Wagener, 2003, Dechert and O'Donnell, 2006, Kiseleva and Wagener, 2010, Kossioris et al., 2008; 2011) but all the work up to now has used a one-dimensional representation of the lake system in terms of the phosphorus accumulation in the water of the lake. However, the basic ecological model (Carpenter, 2005) has two differential equations (and an equation for the loading of phosphorus from the soil around the lake into the water which in a management setting is transformed into the control variable). The other differential equation describes the accumulation of phosphorus in the sediment of the lake (the so-called "mud" equation). The two differential equations interact. Part of the stock of phosphorus in the water ends up in the sediment, and the accumulated phosphorus in the sediment affects the maximum rate of the non-linear feedbacks of phosphorus into the water. The dynamics in the sediment of the lake are found to be much slower than the dynamics in the water of the lake (Janssen and Carpenter, 1999), so that this system has fast-slow dynamics.[2]

The one-dimensional lake management models consider the stock of phosphorus in the sediment as fixed and thus the maximum rate of the non-linear feedbacks into the water as a parameter, and in this way ignore the slow dynamics in the lake. In this paper we will analyze optimal management of the full lake system, with fast-slow dynamics, in which this parameter becomes a slowly changing variable. We will show what the long-run effects are but we will also show that a different type of Skiba point may arise. Such a Skiba point does not indicate indifference between moving to one or the other steady state, as in the one-dimensional representation of the lake, but indicates indifference between one or the other optimal trajectory towards the same long-run steady state. It may happen in such a case, for example, that the optimal trajectory can either move into the oligotrophic area of the lake immediately or can move into the eutrophic area first and flip to the oligotrophic area later, with the same total discounted net benefits.

In this paper we will show what will happen for different values of the parameter that weighs the benefits and the costs, and for different initial values of the stocks of phosphorus in the water and in the sediment of the lake. For a high weight on the loss of ecological services, there is one long-

[2] This depends on the type of the lake. If the lake is a non-thermally stratified shallow lake, then phosphorus release from the sediments may be rapid because the sediments are exposed to waves and thus phosphorous stored in them will be released fast to the surface waters. On the other hand, on thermally stratified deep lakes phosphorous release takes place when stratification breaks down and the lower layer of the lake water is depleted of dissolved oxygen during the year. This means that the rate of phosphorous release is slower in deep lakes than in shallow lakes (Carpenter et al. 1999; Carpenter 2003).

run steady state in the oligotrophic area of the lake but a Skiba manifold of initial points appears from which there are different options for the optimal trajectory towards the steady state. Similarly, for a low weight on the loss of ecological services, there is also one long-run steady state but in the eutrophic area of the lake, with a Skiba manifold of initial points from which there are different options for the optimal trajectory towards the steady state. For intermediate values of the parameter that weighs the benefits and the costs, two stable steady states appear with a Skiba manifold that separates the domains of attraction, but possibly also with a Skiba manifold with different options for optimal trajectories towards the same steady state.

Resources embedded in ecological systems are usually common-pool resources. An important result in the literature on the management of the one-dimensional lake model is that, even if full cooperation would move the lake towards an oligotrophic state, the lake can be trapped in the eutrophic area in the case of more users when initial conditions are above a certain critical point (Mäler et al., 2003). In that analysis, the non-cooperative equilibrium between multiple users of the lake is characterized by the symmetric open-loop Nash equilibrium of this differential game (Başar and Olsder, 1982). We extend this analysis to the full lake system. Full cooperation coincides with optimal management. Furthermore, the trajectories of the open-loop Nash equilibrium can be found by solving the optimal management problem in which the parameter that weighs the benefits and the costs is divided by the number of users. We can now interpret one of the results above in a different way. Since a large number of users corresponds to a low value of the parameter in the optimal management problem, it follows that for sufficiently many users the lake will always become eutrophic, regardless of the initial conditions.

For a lower number of users, whether the lake will end up in an oligotrophic or a eutrophic state depends again on the initial conditions. However, in this case, an area of initial points exists with two possible Nash equilibria, one resulting in an oligotrophic lake and the other one resulting in a eutrophic lake. We show that the first one yields higher net benefits so that it may be argued that the users of the lake will coordinate on this Nash equilibrium, if possible. This also implies that there are no Skiba points in the sense of points of indifference between moving to the oligotrophic or the eutrophic area. However, other switch points arise: moving beyond the upper edge of this area of initial points, the users of the lake have to switch to the Nash equilibrium, resulting in the eutrophic state. This is because the other Nash equilibrium does not exist anymore, in the sense of not being reachable from these initial points. This leads to a substantial drop in welfare. Therefore, this set of initial points can be viewed as a pollution trap, since the users of the lake are trapped in a eutrophic area with a substantial drop in welfare.

This paper also assesses the costs of ignoring the slow dynamics. If the slow variable is considered to be constant, the optimal management strategies can be calculated with the one-dimensional lake system, as in the previous literature. If these strategies are used in a simulation of the two-dimensional lake system, the discounted net benefits can be calculated for the case the slow dynamics are ignored. By comparing this value with the discounted net benefits that result from analyzing the full lake system, the costs of ignoring the slow dynamics are found. We will show that these costs are small. The reason is that the stock of phosphorus in the water does not change much anymore on the slow part of the optimal trajectory so that the net benefits are not strongly affected either.

Section 2 extends the one-dimensional representation of the lake model to the full two-dimensional lake model and discusses the parameter values. Section 3 presents the optimality conditions for optimal management of the lake, and for the symmetric open-loop Nash equilibrium. Sections 4 yields the results for optimal management or full cooperation and Section 5 for the open-loop Nash equilibria. Section 6 calculates the costs of ignoring the slow dynamics, and Section 7 concludes.

2 The Lake Model

The lake model as described by Carpenter (2005) is a system of differential equations for phosphorus density in soil, lake water (denoted by P) and surface sediment (denoted by M for mud). The phosphorus in soil generates run-off into the lake water (denoted by L for loading). The phosphorus densities in the water and the sediment of the lake interact according to the system of differential equations that is given by:

$$\dot{P}(t) = L(t) - (s+h)P(t) + rM(t)f(P(t)), P(0) = P_0 \quad (1)$$

$$\dot{M}(t) = sP(t) - bM(t) - rM(t)f(P(t)), M(0) = M_0 \quad (2)$$

$$f(P) = \frac{P^q}{P^q + m^q} \quad (3)$$

The parameter s denotes the sedimentation rate, h the rate of outflow from the lake system, and b the permanent burial rate. The non-linear term in (1) and (2), given by (3), describes the recycling of phosphorus from the surface sediment into the lake water, and this depends on the level of M, with r denoting the maximum recycling rate. Furthermore, m denotes the level of P where recycling is equal to $0.5r$, and q is a parameter that reflects the steepness of $f(P)$ near m. The lake model in Carpenter (2005) has higher powers in the non-linear term (3) but we take quadratic forms, in order to simplify the analysis and in order to be able to compare the results with previous work that assumed quadratic forms, i.e. $q = 2$. We have found that this simplification does not affect the qualitative structure of the results. The other parameters are set equal to the ones that followed from observations on the watershed of Lake Mendota in Wisconsin, USA: $s = 0.7$, $h = 0.15$, $b = 0.001$, $r = 0.019$ and $m = 2.4$. Janssen and Carpenter (1999) note that the dynamics in M is slow as compared to the dynamics in P. The reason is that small values of the parameters r and b and large values of the parameter q turn M into a slow variable (see Appendix).

The main difference with the model in Carpenter (2005) is that we consider the loading L to be subject to control and thus the result of the optimization of discounted net benefits over an infinite time horizon. The runoff L results from agricultural activities on land and a trade-off occurs between the benefits of agricultural production and the damage of phosphorus accumulation in the water of the lake. Following Mäler et al. (2003), we consider a lake around which $i = 1, ...n$ communities receive two types of benefits: benefits from agricultural production and benefits from the ecosystem services generated by the lake system. Community i, by being able to release phosphorous $L_i(t)$ at time t on the lake, receives agricultural benefits equal to $\ln L_i(t)$. Phosphorous in the lake water $P(t)$ at time t decreases the flow of services generated by the lake ecosystem by $cP(t)^2$. Thus the net flow of benefits accruing to community i is $\ln L_i(t) - cP(t)^2$. The parameter c

indicates the relative importance of the costs versus the benefits. First we will consider the optimal management problem for a single user of the lake, because this forms the basis for all other outcomes. This problem can be written as:

$$\max_{L(\cdot)} \left\{ \int_0^\infty \mathrm{e}^{-\rho t} \left[\ln L(t) - cP(t)^2 \right] dt \right\} \tag{4}$$

subject to (1) and (2).

The traditional analysis of the lake management problem corresponds to the study of only the fast sub-model or:

$$\max_{L(\cdot)} \left\{ \int_0^\infty \mathrm{e}^{-\rho t} \left[\ln L(t) - cP(t)^2 \right] dt \right\}, \tag{5}$$

subject to

$$\dot{P}(t) = L(t) - (s+h) P(t) + rM_0 f(P(t)), P(0) = P_0 \tag{6}$$

$$\dot{M}(t) = 0, M(0) = M_0 \tag{7}$$

Thus, in the traditional analysis of lake management slow dynamics are ignored and the amount of phosphorous in the sediment is assumed to be a fixed parameter at a level M_0, so that equation (2) effectively drops out. By setting $x = P/m$, $a = L/rM_0$ and $\hat{b} = (s+h)m/rM_0$, and by changing the time scale to $(rM_0 t)/m$, equation (6) can be rewritten as $x(t) = a(t) - \hat{b}x(t) + \frac{x(t)^2}{x(t)^2+1}$ (with $q = 2$) which is the basic equation describing the lake dynamics in the traditional analysis of lake management. Note that the change in the time scale implies that the discount rate is changed to $(rM_0\rho)/m$. Mäler et al. (2003) take $\hat{b} = 0.6$, so that the lake has tipping points with hysteresis when moving the fixed loading a up and then down again. The discount rate is set equal to 0.03. The parameter $\hat{b}$ becomes 0.6 for $M_0 = 179$, so that this initial condition for M allows for comparing the results below with the results of the traditional analysis. Furthermore, in the sequel we set the discount rate ρ equal to 0.0425, since we do not change the time scale.

3 Optimal Management, Full Cooperation and Open-Loop Nash Equilibria

The optimal management problem for a single user of the lake, that was defined in the previous section, has the current-value Hamiltonian function:

$$\begin{aligned} &\mathcal{H}\left(L,P,M,\lambda_1,\lambda_2\right) = \ln L - cP^2 + \\ &\lambda_1\left(L-(s+h)\,P + rMf\left(P\right)\right) + \lambda_2\left(sP - bM - rMf\left(P\right)\right) \end{aligned} \tag{8}$$

The necessary conditions resulting from the application of the maximum principle determine the optimal phosphorous loading, $L\left(t\right)$, for interior solutions by:

$$\frac{\partial \mathcal{H}}{\partial L} = \frac{1}{L} + \lambda_1 = 0, \text{ yielding} \tag{9}$$

$$L\left(t\right) = -\frac{1}{\lambda_1\left(t\right)} \tag{10}$$

Setting $f'\left(P\right) = \frac{\partial}{\partial P}\left(\frac{P^q}{P^q+m^q}\right) = \frac{qm^qP^{q-1}}{\left(P^q+m^q\right)^2}$, and dropping t to ease notation, the modified Hamiltonian system becomes:

$$\dot{P} = -\frac{1}{\lambda_1} - (s+h)\,P + rMf\left(P\right) \tag{11}$$

$$\dot{M} = sP - bM - rMf\left(P\right) \tag{12}$$

$$\dot{\lambda}_1 = 2cP + \left[\rho + s + h - rMf'\left(P\right)\right]\lambda_1 - \left[s - rMf'\left(P\right)\right]\lambda_2 \tag{13}$$

$$\dot{\lambda}_2 = (\rho + b + rf\left(P\right))\lambda_2 - rf\left(P\right)\lambda_1 \tag{14}$$

Using (9), $\lambda_1 = \frac{-1}{L}$, so that the time derivative $\dot{\lambda}_1 = \frac{1}{L^2}\dot{L}$, we obtain the modified Hamiltonian system as:

$$\dot{P} = L - (s+h)\,P + rMf\left(P\right) \tag{15}$$

$$\dot{M} = sP - bM - rMf\left(P\right) \tag{16}$$

$$\begin{aligned} \dot{L} = &\left[rMf'\left(P\right) - \rho - s - h\right]L + \\ &\left[2cP - \left[s - rMf'\left(P\right)\right]\lambda_2\right]L^2 \end{aligned} \tag{17}$$

$$\dot{\lambda}_2 = (\rho + b + rf\left(P\right))\,\lambda_2 + \frac{rf(P)}{L} \tag{18}$$

For n communities with the net benefit functions $\ln L_i(t) - cP(t)^2$, $i = 1, 2, ..., n$, as described in the previous section, we can distinguish full cooperation and a non-cooperative Nash equilibrium. When using the maximum principle, we find the so-called open-loop Nash equilibrium (Başar and Olsder, 1982).

In case of full cooperation the lake problem becomes:

$$\max_{L_i(\cdot), i=1,..,n} \left\{ \int_0^\infty e^{-\rho t} \left[\sum_{i=1}^n \ln L_i(t) - ncP(t)^2 \right] dt \right\}, \text{subject to (1) and (2)} \tag{19}$$

Note that symmetry and the logarithmic form in the objective function effectively imply that the full-cooperative solution is independent of the number of communities n, because the problem can be restated as:

$$\max_{L(\cdot)} \left\{ \int_0^\infty e^{-\rho t} n \left[\ln L(t) - cP(t)^2 - \ln n \right] dt \right\}, \text{subject to (1) and (2)} \tag{20}$$

where $L = \sum_{i=1}^n L_i(t)$ denotes the total loading of phosphorous. From this it can be seen that the solution for P, M and for total loading L is the same as for optimal management above, with total loading L as the control variable. It follows that the modified Hamiltonian system under full cooperation is given by (15)-(18) and does not depend on n. However, the resulting total welfare level (20), of course, depends on n.

The open-loop Nash equilibrium is the solution, derived with the maximum principle, of the set of optimal control problems:

$$\max_{L_i(\cdot)} \left\{ \int_0^\infty e^{-\rho t} \left[L_i(t) - cP(t)^2 \right] dt \right\}, i = 1, 2, ..., n, \text{subject to (1) and (2)} \tag{21}$$

We will focus on the symmetric open-loop Nash equilibrium. Each community i maximizes (21) by taking the loadings of all other communities $j \neq i$ as given. The current-value Hamiltonian function for community i becomes:

$$\begin{aligned} &\mathcal{H}(L_i, P, M, \mu_1, \mu_2) = \ln L_i - cP^2 + \\ &\mu_1 [L - (s+h)P + rMf(P)] + \mu_2 [sP - bM - rMf(P)] \end{aligned} \tag{22}$$

The necessary conditions resulting from the application of the maximum principle for the symmetric equilibrium determine phosphorous loading, $L_i(t)$, for interior solutions by:

$$\frac{\partial \mathcal{H}}{\partial L_i} = \frac{1}{L_i} + \mu_1 = 0, \text{ yielding} \tag{23}$$

$$L_i(t) = -\frac{1}{\mu_1(t)}, \text{ for all } i \tag{24}$$

Using (23), $\mu_1 = \frac{-n}{L}$, so that the time derivative $\dot{\mu}_1 = \frac{n}{L^2}\dot{L}$, and setting $\mu_3 = \frac{\mu_2}{n}$,we obtain the modified Hamiltonian system as:

$$\dot{P} = L - (s+h)P + rMf(P) \tag{25}$$

$$\dot{M} = sP - bM - rMf(P) \tag{26}$$

$$\dot{L} = \left[rMf'(P) - \rho - s - h\right]L + \left[\frac{2cP}{n} - \left[s - rMf'(P)\right]\mu_3\right]L^2 \tag{27}$$

$$\dot{\mu}_3 = (\rho + b + rf(P))\mu_3 + \frac{rf(P)}{L} \tag{28}$$

which is exactly the same system as (15)-(18), except for the term $2cP/n$ in (27). It follows that the set of possible trajectories of the symmetric open-loop Nash equilibrium can be found by solving the modified Hamiltonian system of the optimal management problem with relative cost parameter c/n instead of c. Note, however, that these are only necessary conditions and that the individual welfare indicators (21) are different. This implies that the Nash equilibrium trajectories for some c and n may differ from the optimal trajectories for c/n. Moreover, multiple Nash equilibria may exist. This will become clear in Section 5.

Summarizing, we can study the management of the lake by studying the modified Hamiltonian system (15)-(18) and by varying the cost parameter c that reflects the relative importance of the costs versus the benefits. We can vary c, but we can also fix c at some value and then divide c by the number of communities n in order to move from the optimal management or full-cooperative outcome to the symmetric open-loop Nash equilibrium. By implementing optimality or the Nash equilibrium conditions we can identify the optimal and Nash equilibrium trajectories.

We can compare the results with the results for the one-dimensional representation of the lake system by taking $c = 1/m^2 = 0.1736$. This corresponds to $c = 1$ in the analysis of Mäler et al. (2003) who use the single lake equation as described in the previous section, with the transformation $x = P/m$. Mäler et al. (2003) find one saddle-point stable steady state in the oligotrophic area of the lake under full cooperation. In the open-loop Nash equilibrium they find two saddle-point stable steady states, one in the oligotrophic area and one in the eutrophic area of the lake, and an unstable steady state in between. More specifically, using the one-dimensional representation of the lake, with a fixed parameter M_0, the steady states are given by the solutions to the set of equations:

$$0 = L - (s + h) P + rM_0 f (P) \quad (29)$$

$$0 = \left[rM_0 f' (P) - \rho - s - h)\right] + 2cPL \quad (30)$$

For the parameter values given in section 2, with $c = 0.1736$, the full-cooperative outcome has only one saddle-point stable steady state $P^C = 0.84793$ in the oligotrophic area of the lake (the superscript C denotes co-operation). The open-loop Nash equilibrium for two communities has two saddle-point stable steady states $P^N_{21} = 0.94318$ and $P^N_{22} = 3.8023$ (the super-script N denotes Nash and the first subscript 2 denotes two communities). Depending on the initial condition of the lake, the open-loop Nash equilibrium trajectory will either move to P^N_{21} in the oligotrophic area of the lake or to P^N_{22} in the eutrophic area of the lake. The question is what happens in the full lake model, with fast-slow dynamics. This will be investigated in the next sections.

4 Results for Optimal Management or Full Cooperation

Using the full lake model (1)-(3), in which the phosphorus density in the surface sediment changes, we have to solve the modified Hamiltonian system (15)-(18), with fast-slow dynamics. First we will look at the optimal management problem for three values of the relative cost parameter: $c_1 = 0.1736$, $c_2 = 0.0868$ and $c_3 = 0.057867$. The solutions coincide with the full-cooperative outcomes for these parameter values, because the modified Hamiltonian systems are the same and because the welfare levels are the same (up to a constant depending on n) so that the Skiba points are the same. The solutions for $c_2 = 0.0868$ and $c_3 = 0.057867$ are also the basis for the open-loop Nash equilibria for $c = 0.1736$ with two and three communities, respectively, because the modified Hamiltonian systems are the same, but the final outcomes will differ, of course. We will look at the open-loop Nash equilibria in the next section.

The optimal control of the two-dimensional system yields a four-dimensional modified Hamiltonian system but the fast-slow dynamics are helpful in developing the solution algorithm. The system adjusts quickly on the fast dimensions and then the trajectory moves slowly with the changes on the slow dimensions. We use the package OCMat[3] which is an extension of the Matlab solver Bvp4c for these boundary value problems.

[Insert Figure 1 here]

Figure 1 presents the projection of the full-cooperative outcome for $c_1 = 0.1736$ on the (P, M)-plane for the phosphorus densities in the lake water P and in the surface sediment M. All the trajectories converge to $(P_1^C, M_1^C) = (0.77420, 194.19)$ (i.e. the green point) in the oligotrophic area of the lake (the subscript 1 refers to c_1). It is the intersection point of the isoclines $\dot{P} = 0$ (the dashed grey curves) and $\dot{M} = 0$ (the grey curve) under optimal loading L^*. The phosphorus density in the lake water becomes somewhat lower than in the one-dimensional analysis of the lake. Starting for example in $(P_0, M_0) = (0, 179)$ or in $(P_0, M_0) = (5.5, 179)$, there is first a relatively fast adjustment of P to approximately the steady-state value of P in the one-dimensional analysis of the lake, followed by a slow adjustment of M upwards and P downwards (along the optimal isocline $\dot{P} = 0$). This is not surprising.

However, starting in higher values of M_0, a manifold of a different type of Skiba points occurs (the black curve). Those Skiba points do not separate different domains of attraction towards different steady states but separate different optimal trajectories towards the same long-run steady state. More specifically, starting in the diamond on that Skiba manifold yields two optimal trajectories towards the steady state, with the same total discounted net benefits. The first one moves into the oligotrophic area of the lake immediately and then slowly down towards the long-run steady state (along a part of the optimal isocline $\dot{P} = 0$). The other one moves fast into the eutrophic area of the lake first (the dashed blue line), then slowly down and in the

[3] See http://orcos.tuwien.ac.at/research/ocmat_software

direction of the oligotrophic area of the lake (along another part of the optimal isocline $\dot{P} = 0$), and finally quickly towards the long-run steady state. Furthermore, starting in $(P_0, M_0) = (0, 240)$ or in $(P_0, M_0) = (5.5, 240)$, there is first a relatively fast adjustment of P, followed by a slow adjustment of M and P towards the branches of the two optimal trajectories just described. Finally, starting at the end of the black Skiba manifold, in $(P_0, M_0) = (2.7376, 206.56)$, there is only one optimal trajectory towards the long-run steady state (the red curve). Actually, starting anywhere below that point (for example in $(P_0, M_0) = (5.5, 200)$), only one optimal trajectory occurs.

[Insert Figure 2 here]

Figure 2 presents the projection of the full-cooperative outcome for $c_2 = 0.0868$ on the (P, M)-plane for the phosphorus densities in the lake water P and in the surface sediment M. This case has two possible long-run steady states, namely $(P^C_{21}, M^C_{21}) = (0.87017, 189.90)$ and $(P^C_{22}, M^C_{22}) = (3.3551, 173.09)$ (the green points), one in the oligotrophic area and one in the eutrophic area of the lake, with a traditional Skiba manifold separating the domains of attraction (the black curve) (the first subscript 2 refers to c_2). The green points are the intersection points of the isoclines $\dot{P} = 0$ (the dashed grey curves) and $\dot{M} = 0$ (the grey curve) under optimal loading L^*. The phosphorus density in the lake water becomes again somewhat lower than in the one-dimensional analysis of the lake. Starting in $(P_0, M_0) = (0, 179)$, there is first a relatively fast adjustment of P to approximately the oligotrophic steady-state value of P in the one-dimensional analysis of the lake, followed by a slow adjustment of M upwards and P downwards (along the optimal isocline $\dot{P} = 0$). Starting in $(P_0, M_0) = (5.5, 179)$, there is first a relatively fast adjustment of P to approximately the eutrophic steady-state value of P in the one-dimensional analysis of the lake, followed by a slow adjustment of M and P downwards (along the optimal isocline $\dot{P} = 0$). Furthermore, starting in either one of the diamonds on the Skiba manifold, there are two optimal trajectories, one towards the oligotrophic long-run steady state and one towards the eutrophic long-run steady state, with the same total discounted net benefits. Starting in $(P_0, M_0) = (0, 240)$ or in $(P_0, M_0) = (5.5, 240)$ basically yields the same story as for $M_0 = 179$. However, if the initial phosphorus density in the surface sediment M_0 is lower than 155.84, all the optimal trajectories converge to the eutrophic steady state $(P_{22}^C, M_{22}^C) = (3.3551, 173.09)$ (but see Figure 4 with explanations below).

[Insert Figure 3 here]

Figure 3 presents the projection of the full-cooperative outcome for $c_3 = 0.057867$ on the (P, M)-plane for the phosphorus densities in the lake water P and in the surface sediment M. All the optimal trajectories converge to $(P_3^C, M_3^C) = (4.7957, 207.29)$ (the green point) in the eutrophic area of the lake (the subscript 3 refers to c_3). It is the intersection point of the isoclines $\dot{P} = 0$ (the dashed grey curves) and $\dot{M} = 0$ (the grey curve) under optimal loading L^*. Starting in $(P_0, M_0) = (0, 179)$ or in $(P_0, M_0) = (5.5, 179)$, there is first a relatively fast adjustment of P, followed by a slow adjustment of M upwards and P upwards (along the optimal isocline $\dot{P} = 0$). Starting in higher values of M_0, a manifold of a different type of Skiba points occurs again (the black curve). Those Skiba points do not separate different domains of attraction towards different steady states but separate different optimal trajectories towards the same long-run steady state. More specifically, starting in the diamond on that Skiba manifold yields two optimal trajectories towards the steady state, with the same total discounted net benefits. The first trajectory moves into the eutrophic area of the lake immediately and then slowly down towards the long-run steady state (along the optimal isocline $\dot{P} = 0$). The other trajectory moves fast into the oligotrophic area of the lake first (the dashed blue line), then slowly down (along a part of the optimal isocline $\dot{P} = 0$), then fast again into the eutrophic area of the lake, and finally slowly down again towards the long-run steady state (along another part of the optimal isocline $\dot{P} = 0$). Furthermore, starting in $(P_0, M_0) = (0, 240)$ or in $(P_0, M_0) = (5.5, 240)$, the story is basically the same as above. Finally, starting at the end of the black Skiba manifold there is only one optimal trajectory towards the long-run steady state (the red curve). Actually, starting anywhere below that point, there is again only one optimal trajectory.

The general picture is clear. The parameter c reflects the relative importance of the costs versus the benefits of phosphorus loadings. If c is high, it is optimal to end up in the oligotrophic area of the lake with a high level of ecological services and if c is low, it is optimal to end up in the eutrophic area of the lake with a low level of ecological services. For intermediate values of c, the outcome depends on the initial conditions of the lake, which implies that Skiba points occur where one is indifferent between ending up in the oligotrophic or in the eutrophic area of the lake. Figures 1-3 show three characteristic cases but it is interesting to consider the transition from one case to the other.

[Insert Figure 4 here]

Figure 4 presents the projection of the full-cooperative outcome for $c = 0.092152$ on the (P, M)-plane for the phosphorus densities in the lake water P and in the surface sediment M. This picture arises as follows. When we gradually decrease the value of the relative cost parameter c from $c_1 = 0.1736$ to $c_2 = 0.0868$, first the second possible long-run equilibrium point in the eutrophic area of the lake appears, with a Skiba manifold separating the domains of attraction. This Skiba manifold (the black curve in Figure 4) bends to the right below, so that all the optimal trajectories for low initial values M_0 still converge to the oligotrophic steady state. Then a second type of Skiba manifold appears (the grey curve in Figure 4). These are points from where two equivalent optimal trajectories originate that converge to the same oligotrophic steady state. This leads to the situation depicted in Figure 4. Area I indicates the domain of attraction of the eutrophic long-run steady state and areas IIa and IIb indicate the domains of attraction of the oligotrophic long-run steady state, separated by this second type of Skiba manifold. Decreasing the value of the relative cost parameter c just a little bit further, the outcome bifurcates to the situation depicted in Figure 2, with only the Skiba manifold that is separating the domains of attraction. As we have seen, this Skiba manifold now bends to the left below, so that all the optimal trajectories for low initial values M_0 converge to the eutrophic steady state. When we gradually decrease the value of the relative cost parameter c from $c_2 = 0.0868$ to $c_3 = 0.057867$, at some point the long-run equilibrium in the oligotrophic area of the lake disappears and the outcome becomes as depicted in Figure 3.

5 Results for Open-Loop Nash Equilibria

It was shown in Section 3 that for the relative cost parameter $c = 0.1736$ and the number of communities $n = 2$ we get the same modified Hamiltonian system for the open-loop Nash equilibrium as for optimal management with $c_2 = 0.0868$. This implies that we get the same two possible long-run steady states $(P_{21}^N, M_{21}^N) = (P_{21}^C, M_{21}^C) = (0.87017, 189.90)$ and $(P_{22}^N, M_{22}^N) = (P_{22}^C, M_{22}^C) = (3.3551, 173.09)$ as in Figure 2. However, the Nash equilibrium trajectories, driven by the individual welfare indicators (21), differ from the optimal trajectories.

[Insert Figure 5 here]

Figure 5 presents the projection of the Nash equilibrium trajectories on the (P, M)-plane for the phosphorus densities in the lake water P and in the surface sediment M, starting in initial conditions on a line connecting the two long-run steady states. On the upper part of that line, only one Nash equilibrium exists, leading to the oligotrophic long-run steady-state (the dashed blue lines). On the lower part of that line, also only one Nash equilibrium exists, leading to the eutrophic long-run steady-state (the solid blue lines). However, in the middle part of that line, two Nash equilibria exist, one leading to the oligotrophic steady state and one leading to the eutrophic steady state. The patterns are the same as for the optimal trajectories in Section 4: first a fast adjustment of P, followed by a slow adjustment of M and P.

[Insert Figure 6 here]

Figure 6 is similar to Figure 5 but now for initial conditions with a fixed $M_0 = 179$. The initial points with two Nash equilibria in both Figure 5 and Figure 6 do not have to be Skiba indifference points. Although we focus on symmetric Nash equilibria, so that the welfare levels for the two communities are the same in each Nash equilibrium, these welfare levels may be different when we compare the two Nash equilibria.

[Insert Figure 7 here]

Figure 7 depicts the resulting discounted net benefits (21) for the Nash equilibrium trajectories in Figure 6 as a function of the initial condition P_0. It shows that when two Nash equilibria exist, the one leading to the oligotrophic steady state has higher welfare everywhere. We may argue that the two communities will try to coordinate on the best Nash equilibrium but then an interesting phenomenon occurs. When we increase the initial condition P_0 beyond the point where two Nash equilibria still exist, the communities cannot coordinate on this best Nash equilibrium anymore, because it does not exist anymore. They have to switch to the bad Nash equilibrium, leading to the eutrophic long-run steady state. The drop in welfare at this point is considerable (from -57.06 to -73.32). This point can be viewed as the beginning of a pollution trap. Moving beyond this point implies that the good Nash equilibrium is not reachable any more and that the lake will end up in a eutrophic state with a considerable drop in welfare. This is because, due to open-loop strategies, users commit to their optimal path given the specific initial condition. If the initial condition is such that the oligotrophic steady state is not reachable, then users are committed to a path that traps them into the eutrophic region with a considerable drop in welfare.

6 Welfare Loss Ignoring Slow Dynamics

In this section we return to optimal management but now we consider what happens if the slow dynamics of the phosphorus accumulation in the surface sediment M is ignored. This means that we implement the optimal loading that is found with the one-dimensional representation of the lake into the two-dimensional lake system, and compare the result with the optimal management of the two-dimensional lake system.

Figure 1 presents the optimal management of the two-dimensional lake system for relative cost parameter $c_1 = 0.1736$. As we have seen, for the initial conditions $(P_0, M_0) = (0, 179)$ or $(P_0, M_0) = (5.5, 179)$ there is first a relatively fast adjustment of P, followed by a slow adjustment of M upwards and P downwards (along the optimal isocline $\dot{P} = 0$). If we now implement the optimal loading that is found with the one-dimensional representation of the lake into the two-dimensional lake system, we get Figure 8.

[Insert Figure 8 here]

Figure 8 shows that ignoring the slow dynamics in M leads to a cyclical pattern in P and M. Instead of converging to the long-run steady state $(P_1^C, M_1^C) = (0.77420, 194.19)$ as in Figure 1, P starts cycling at some point in time between about 0.7 and 2.25 and M starts cycling between about 170 and 181.5. This is not optimal, of course, but it also not dramatic. Starting in $(P_0, M_0) = (0, 179)$, the optimal welfare is equal to -26.62 and the welfare ignoring the slow dynamics in M is equal to -26.68. Starting in $(P_0, M_0) = (5.5, 179)$, the optimal welfare is equal to -49.49 and the welfare ignoring the slow dynamics in M is equal to -49.52. The reason why the welfare loss is small is that the fast adjustment in P already moves the lake into the oligotrophic area, close to the optimal long-run steady state, before the changes in M start to have an effect. Ignoring the slow dynamics in M

will not yield convergence to the optimal point but it will not switch the lake to the eutrophic area either.

Figure 2 presents the optimal management of the two-dimensional lake system for relative cost parameter $c_2 = 0.0868$. As we have seen, in this case there are two optimal long-run steady states, $(P_{21}^C, M_{21}^C) = (0.87017, 189.90)$ and $(P_{22}^C, M_{22}^C) = (3.3551, 173.09)$, one in the oligotrophic area and one in the eutrophic area of the lake, with a Skiba manifold separating the domains of attraction. For initial condition $(P_0, M_0) = (0, 179)$ there is first a relatively fast adjustment of P, followed by a slow adjustment of M upwards and P downwards (along the optimal isocline $\dot{P} = 0$), and for initial condition $(P_0, M_0) = (5.5, 179)$ there is first a relatively fast adjustment of P, followed by a slow adjustment of M and P downwards (along the optimal isocline $\dot{P} = 0$). Furthermore, starting in the lower diamond on the Skiba manifold, there are two optimal trajectories, one towards the oligotrophic long-run steady state and one towards the eutrophic long-run steady state, with the same total discounted net benefits. If we now implement the optimal loading that is found with the one-dimensional representation of the lake into the two-dimensional lake system, we get Figure 9.

[Insert Figure 9]

Figure 9 shows that ignoring the slow dynamics in M leads to a cyclical pattern in P and M in the oligotrophic area of the lake, but it leads to convergence in the eutrophic area of the lake. Note that the diamond is not a Skiba point anymore, because now we are not implementing the optimal loading for the two-dimensional lake system. In the oligotrophic area, instead of converging to the long-run steady state (P_{11}^C, M_{11}^C) as in Figure 2, P starts cycling at some point in time between about 0.75 and 2.25 and M starts cycling between about 170 and 180.5. Starting in $(P_0, M_0) = (0, 179)$, the optimal welfare is equal to -25.13 and the welfare ignoring the slow dynamics in M is equal to -25.23. In the eutrophic area, instead of converging to the long-run steady state (P_{22}^C, M_{22}^C) as in Figure 2, the lake system will now converge to the long-run steady state $(P, M) = (4.0744, 188.81)$. Starting in $(P_0, M_0) = (5.5, 179)$, the optimal welfare is equal to -37.35 and the welfare ignoring the slow dynamics in M is equal to -37.73. This welfare loss is a bit higher than in the previous cases because the trajectory differs more from the optimal trajectory, but the welfare loss remains to be small because the trajectory remains in the eutrophic area. Ignoring the slow dynamics in M does not lead to a large loss in welfare in this case either.

Figure 3 presents the optimal management of the two-dimensional lake system for relative cost parameter $c_3 = 0.057867$. As we have seen, for the initial conditions $(P_0, M_0) = (0, 179)$ or $(P_0, M_0) = (5.5, 179)$ there is first a relatively fast adjustment of P, followed by a slow adjustment of M upwards and P upwards (along the optimal isocline $\dot{P} = 0$). If we now implement the optimal loading that is found with the one-dimensional representation of the lake into the two-dimensional lake system, we get Figure 10.

[Insert Figure 10 here]

Figure 10 shows that ignoring the slow dynamics in M leads to convergence again but towards a different long-run steady state. Instead of converging to the long-run steady state $(P_3^C, M_3^C) = (4.7957, 207.29)$ as in Figure 3, the lake system will now converge to the long-run steady state $(P, M) = (6.3858, 253.28)$. Starting in $(P_0, M_0) = (0, 179)$, the optimal welfare is equal to -20.29 and the welfare ignoring the slow dynamics in M is equal to -20.76. Starting in $(P_0, M_0) = (5.5, 179)$, the optimal welfare is equal to -26.33 and the welfare ignoring the slow dynamics in M is equal to -26.90. The welfare loss is again a bit higher than in the previous cases because the trajectory moves to a different long-run steady state, but the welfare loss remains to be small because the trajectory remains in the eutrophic area. Ignoring the slow dynamics in M does not lead to a large loss in welfare in this case either.

Summarizing, even though the one-dimensional representation of the lake ignores the slow dynamics in the phosphorus accumulation in the surface sediment of the lake, optimal management with this simplified model does not perform so badly. Implementing the one-dimensional optimal loading of phosphorus into the full lake system and comparing the resulting welfare with the optimal welfare shows that the welfare loss is small. The reason is in fact the slow dynamics. Optimal management of the simplified model already makes the choice to move the lake either into the oligotrophic or into the eutrophic area. Further adjustments that are needed for optimal management of the full lake system prove to be relatively unimportant.

7 Conclusion

The lake is a ecological system providing services that can be damaged by phosphorus loadings resulting from profitable human activities. This paper considers optimal management and Nash equilibria in the analysis of the full lake model and compares the results with the previous literature that has focused on a simplified version of the model. The full model includes, besides the damaging accumulation of phosphorus in the water, the slow accumulation of phosphorus in the surface sediment describing the process of sedimentation of phosphorus and recycling back into the water.

The main pattern of the results stays intact. If a high value is attached to the ecological services as compared to the other profitable activities, optimal management will move the lake towards an oligotrophic state with a high level of ecological services. If a low value is attached to the ecological services, optimal management will move the lake towards a eutrophic state with a low level of ecological services. For intermediate values, Skiba indifference points exist with optimal trajectories to either an oligotrophic or a eutrophic state. However, optimal management of the more complicated two-dimensional non-linear system gives rise to another type of Skiba points. Starting in these points, different optimal trajectories can be chosen, leading to the same long-run steady state, with the same level of welfare. Typically, these trajectories either move to the targeted area of the lake directly or move to the other area first and stay there for some time, before moving to the targeted area.

Another interesting result was found in case of common property, with a number of communities who are damaging the lake but also using the common ecological services. Non-cooperative behavior is characterized with an open-loop Nash equilibrium of this differential game. An area of initial points exist with two possible Nash equilibria, one leading to an oligotrophic steady state and the other one leading to a eutrophic steady state. However, these initial points are not Skiba points because the welfare levels are different in the two Nash equilibria. In fact, the Nash equilibrium moving towards the oligotrophic state always dominates the Nash equilibrium moving towards the eutrophic state. It can be argued that the communities will coordinate on the good Nash equilibrium, but then a potential pollution trap exists where the communities have to switch to the bad Nash equilibrium.

The good one does not exist anymore in the sense that it is not reachable from the specific set of initial conditions. Being in the pollution trap implies a substantial drop in welfare.

An important question is how much welfare is lost when the optimal phosphorus loadings found with the simplified version of the lake model are implemented in the more complicated version. It is shown that optimal convergence of the trajectories may turn into cyclical behavior or into convergence towards a different long-run steady state, but the welfare losses are small. The reason is that the process of sedimentation and recycling back into the water is slow as compared to the phosphorus accumulation in the water. Therefore, optimal management ignoring the slow process will already move the lake into the right area. The adjustments that are needed for optimality when the slow changes kick in will not affect welfare very much. In the case of the lake, simplified optimal management will not perform very badly.

The lake model is a metaphor for many of the environmental problems facing the world today. This paper provides insights and analytical tools for developing optimal management of the lake and characterizing common-property outcomes. These systems are complicated because they are non-linear and because some parameters are slowly changing. This paper shows how these systems can be analyzed and managed properly.

8 Appendix

For $b = 0$ the steady state of the system (1)-(3) is given by:

$$(\hat{P}, \hat{M}) = \left(\frac{\hat{L}}{h}, \frac{s\hat{L}}{rh} \left(1 + \frac{(hm)^q}{\hat{L}^q} \right) \right) \tag{31}$$

The Jacobian $\hat{J}$ becomes:

$$\hat{J} = \begin{bmatrix} -(s+h) + r\hat{M}f'(\hat{P}) & rf(\hat{P}) \\ s - r\hat{M}f'(\hat{P}) & -rf(\hat{P}) \end{bmatrix} \tag{32}$$

so that the determinant is given by:

$$\det \hat{J} = hrf(\hat{P}) = \frac{hr}{1 + \tilde{m}^q}, \tilde{m} = \frac{hm}{\hat{L}} \tag{33}$$

For $r = 0$ the determinant of the Jacobian $\hat{J}$ is zero, so that one of the eigenvalues of the Jacobian $\hat{J}$ is zero with eigenvector [0 1]'. This implies that for small values of r and large values of q one of the eigenvalues of the Jacobian $\hat{J}$ is close to zero with an eigenvector close to [0 1]', so that the second variable M is a slow variable. This will still be the case for small values of b. Q.E.D.

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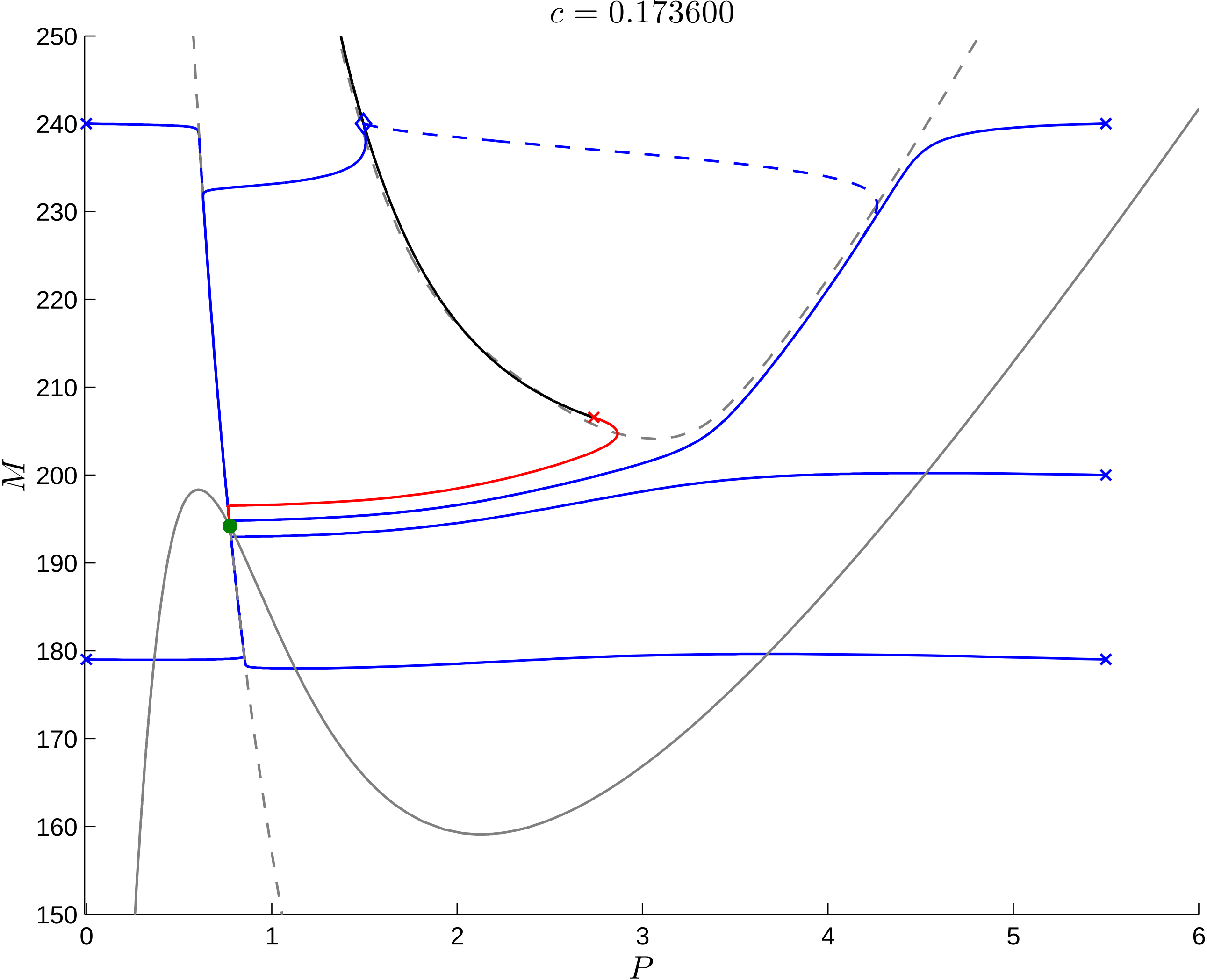

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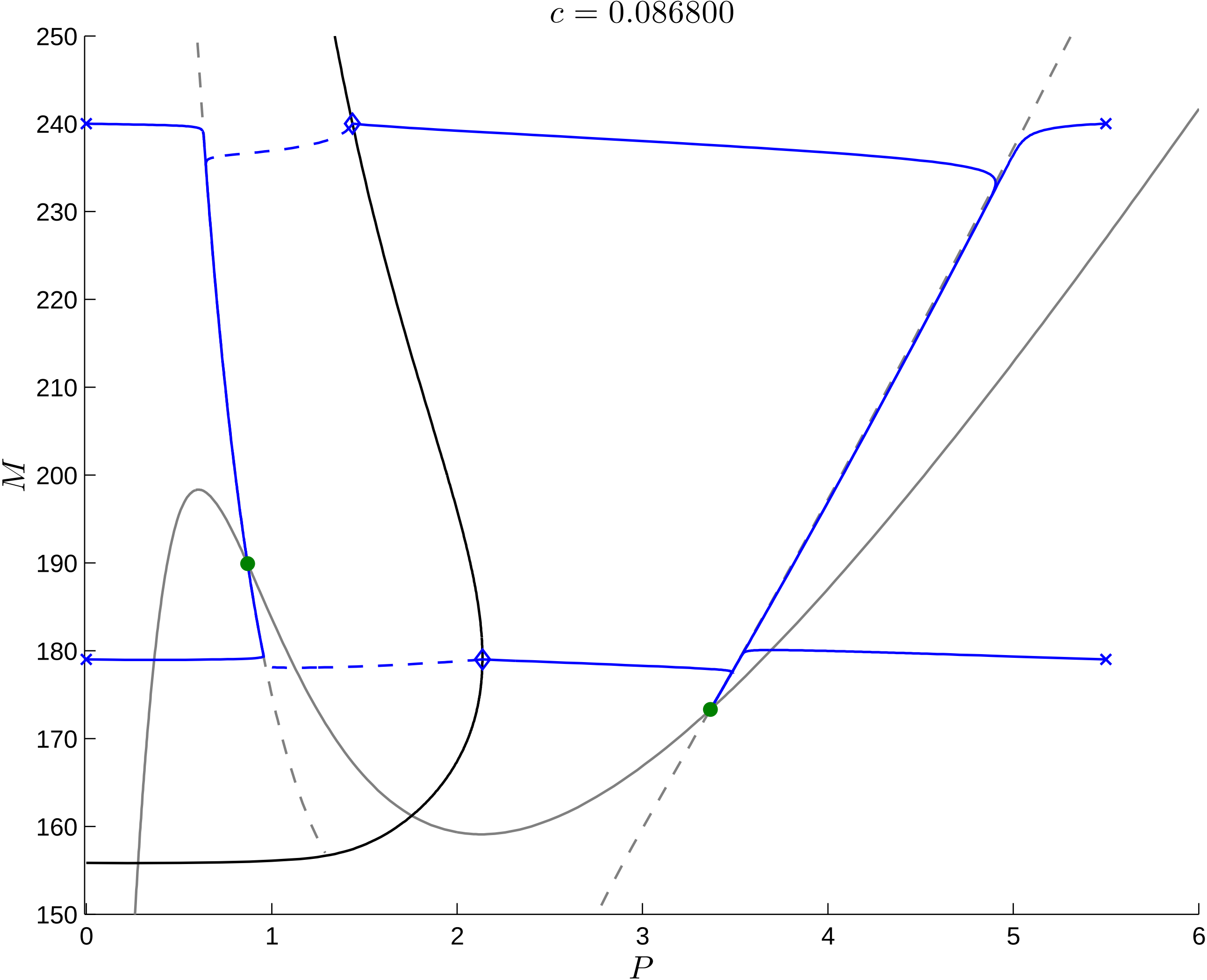

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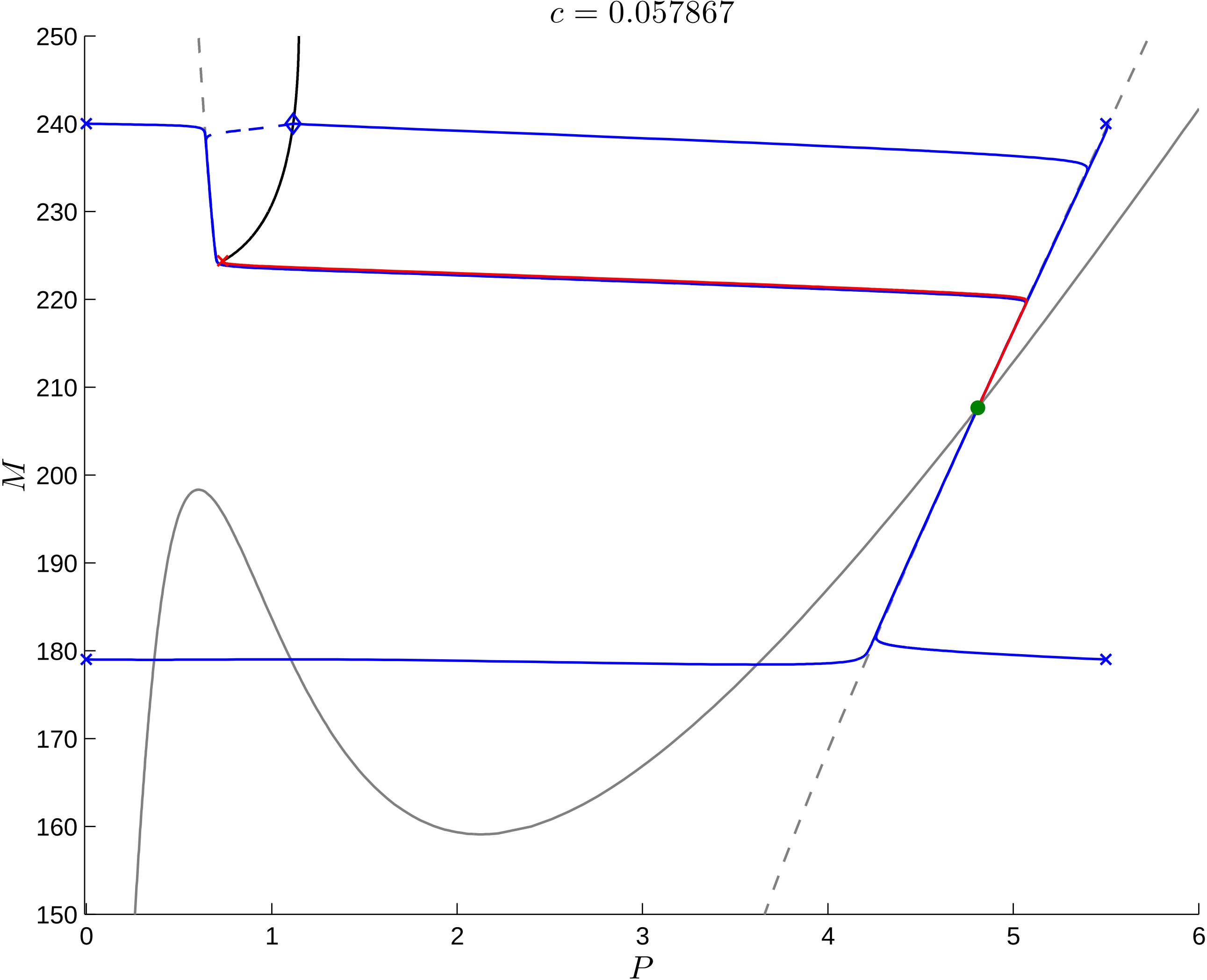

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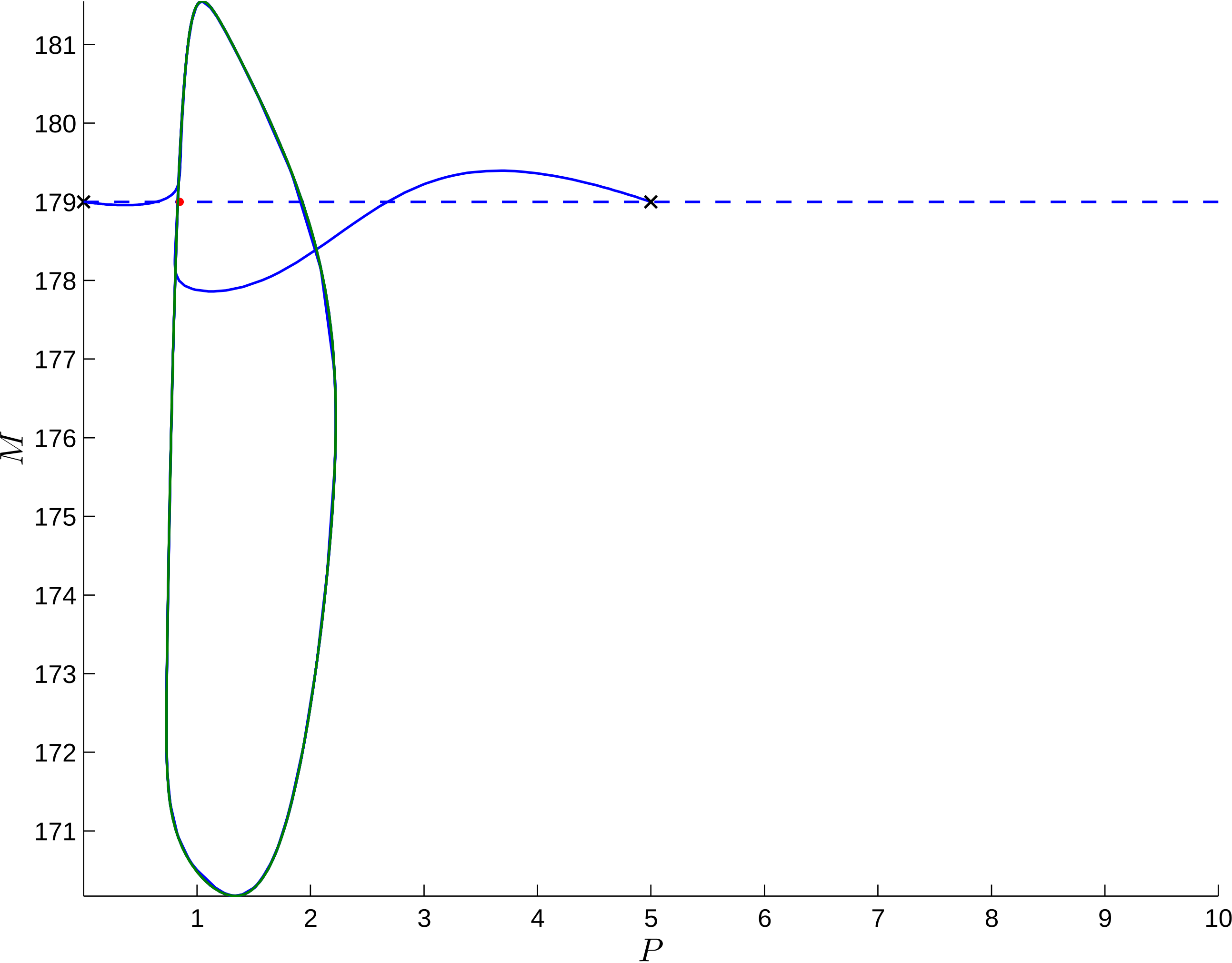

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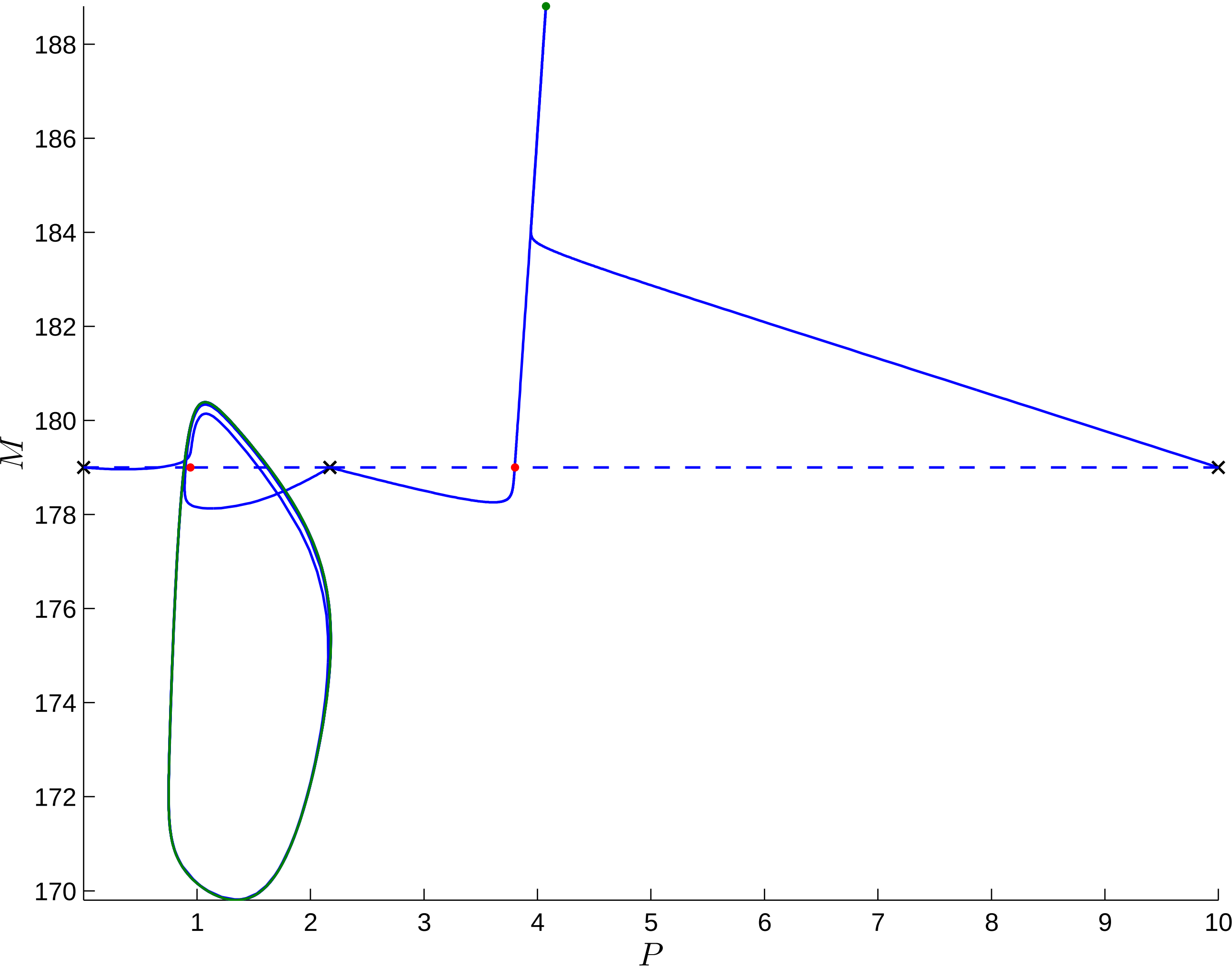

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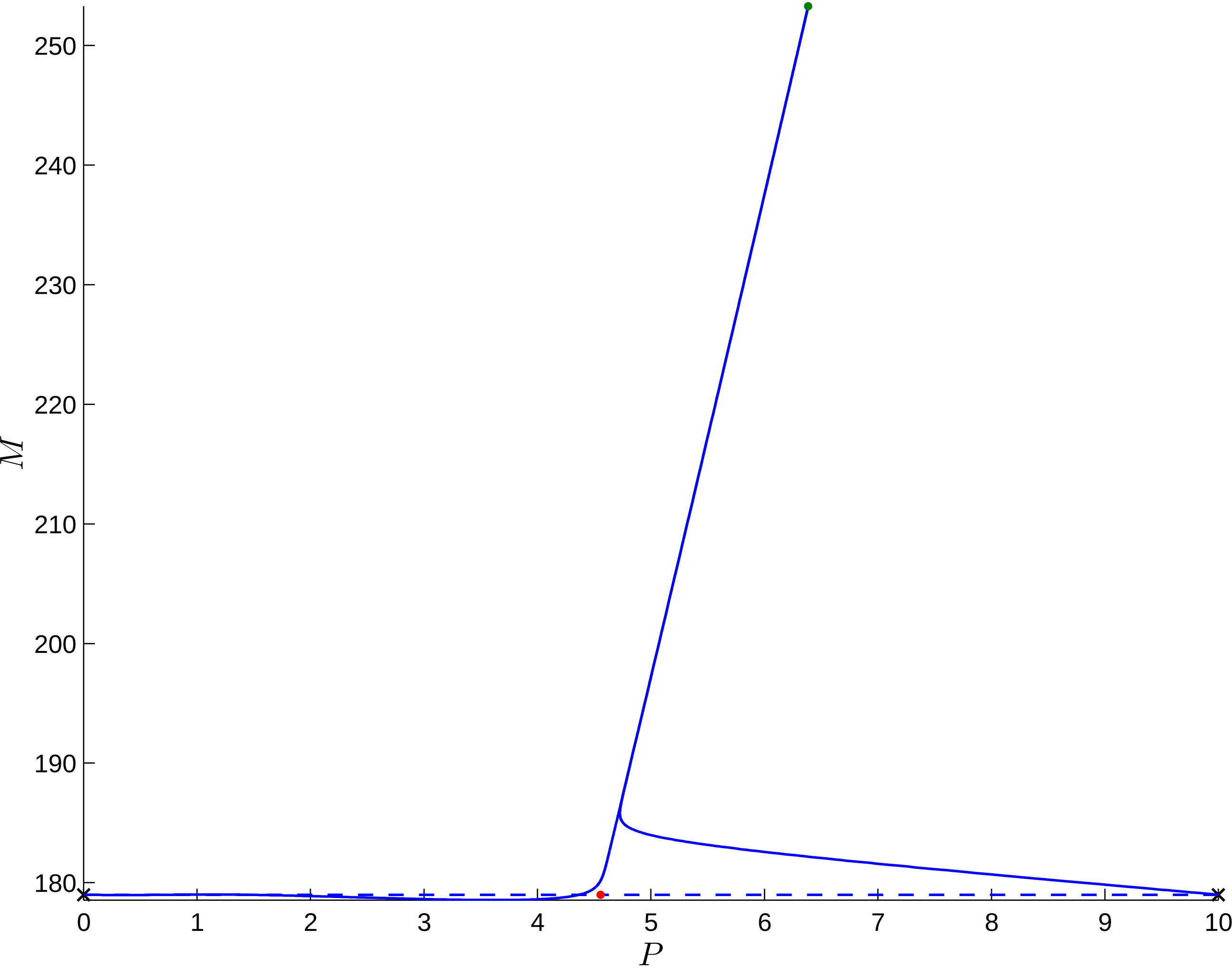

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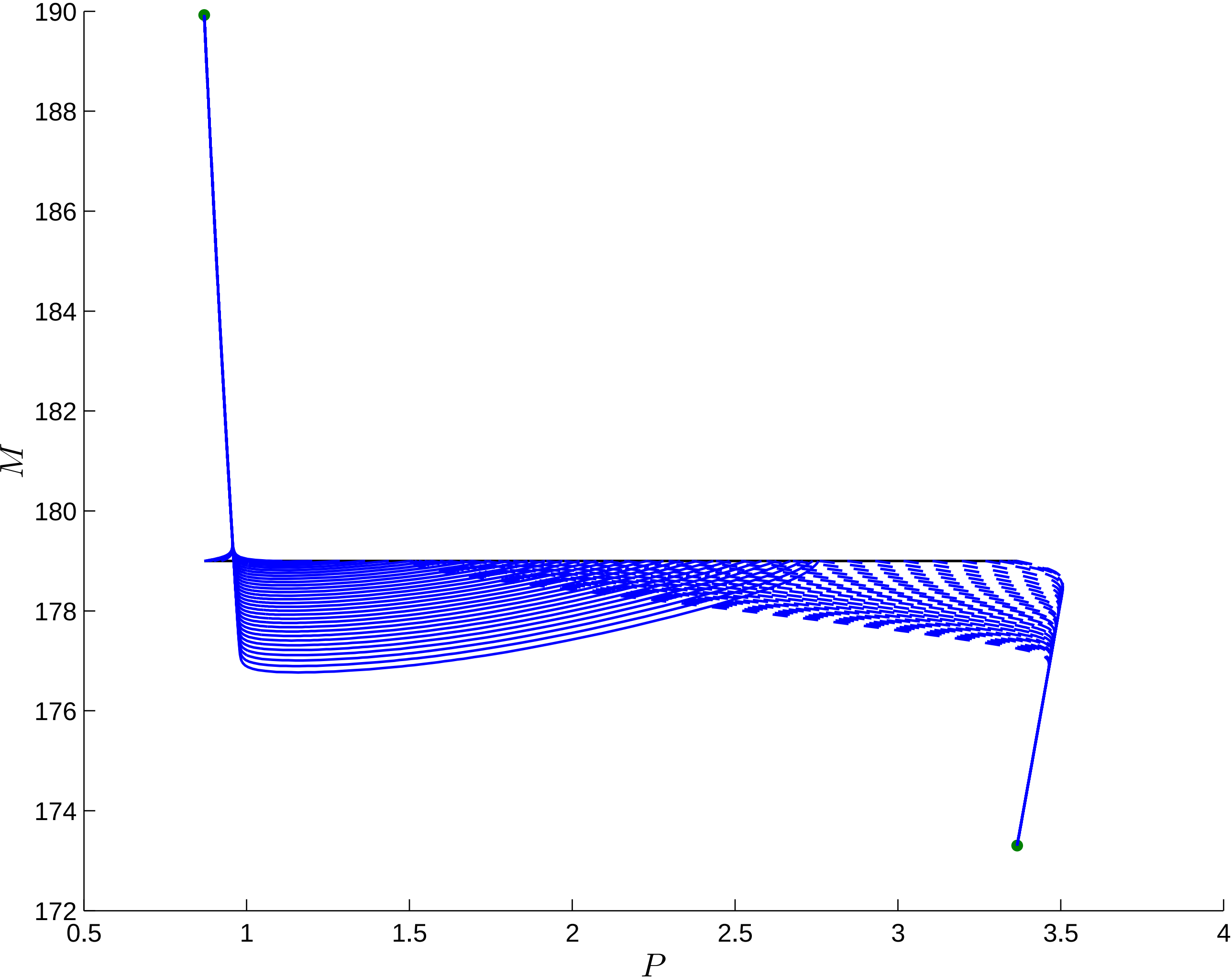
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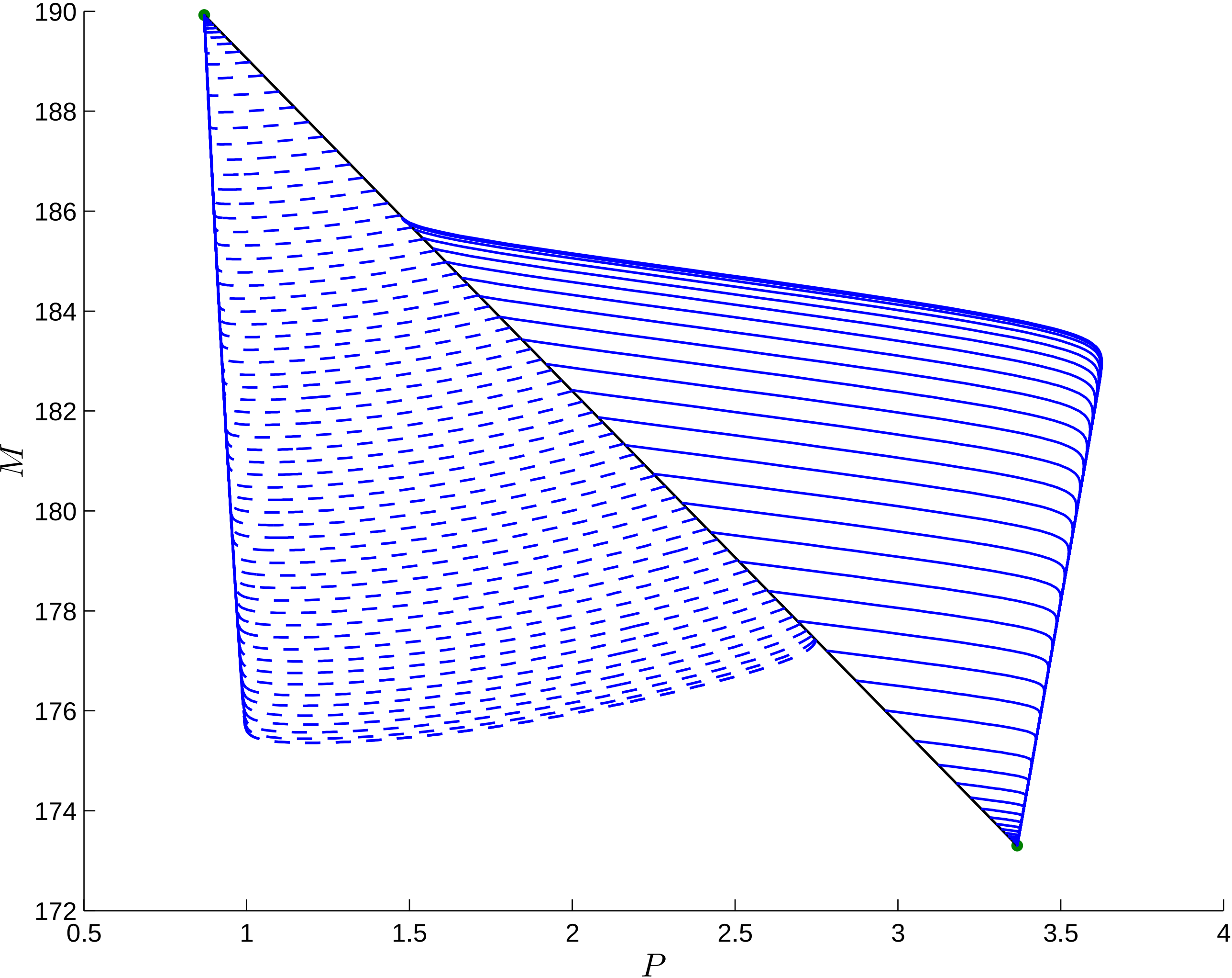
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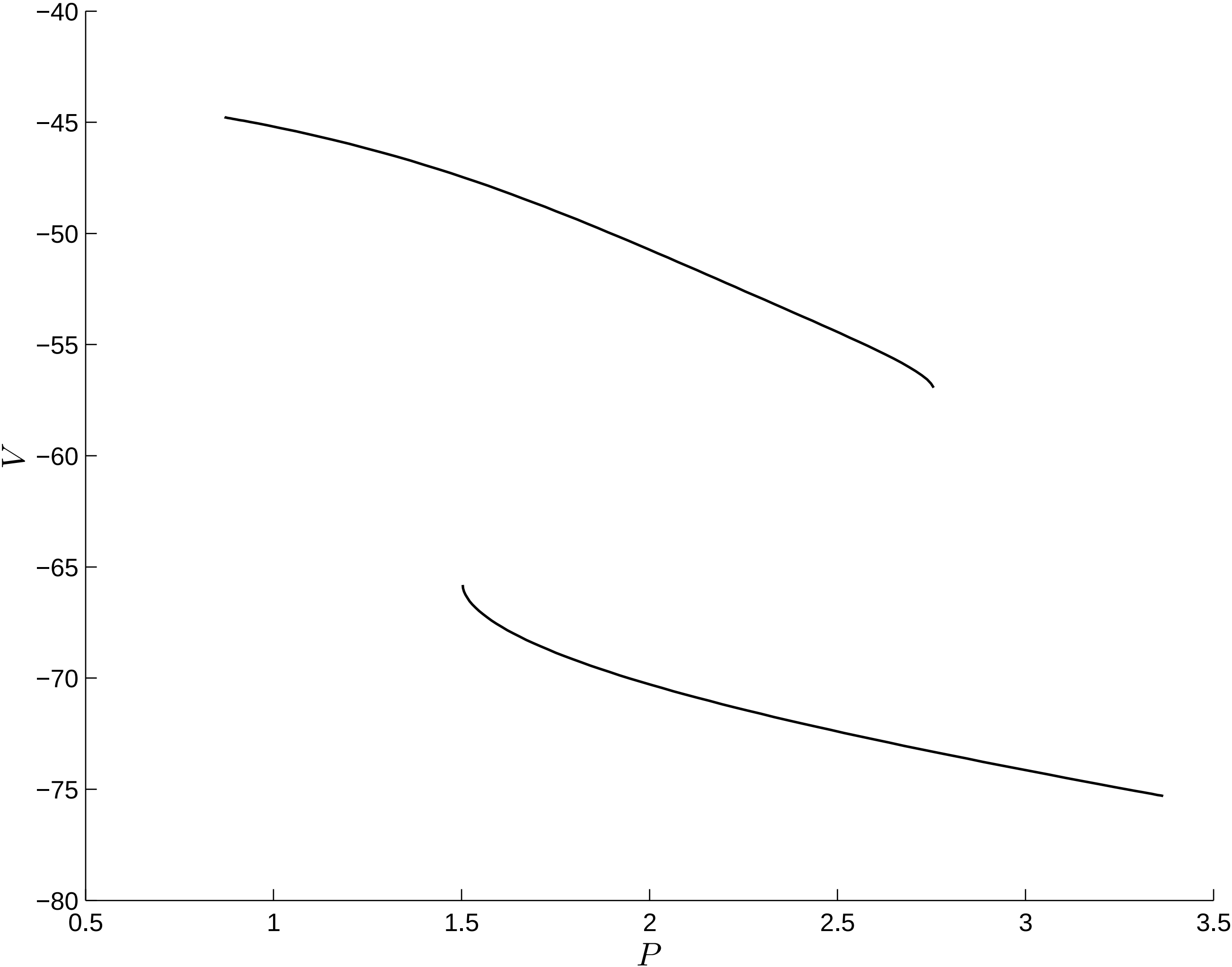

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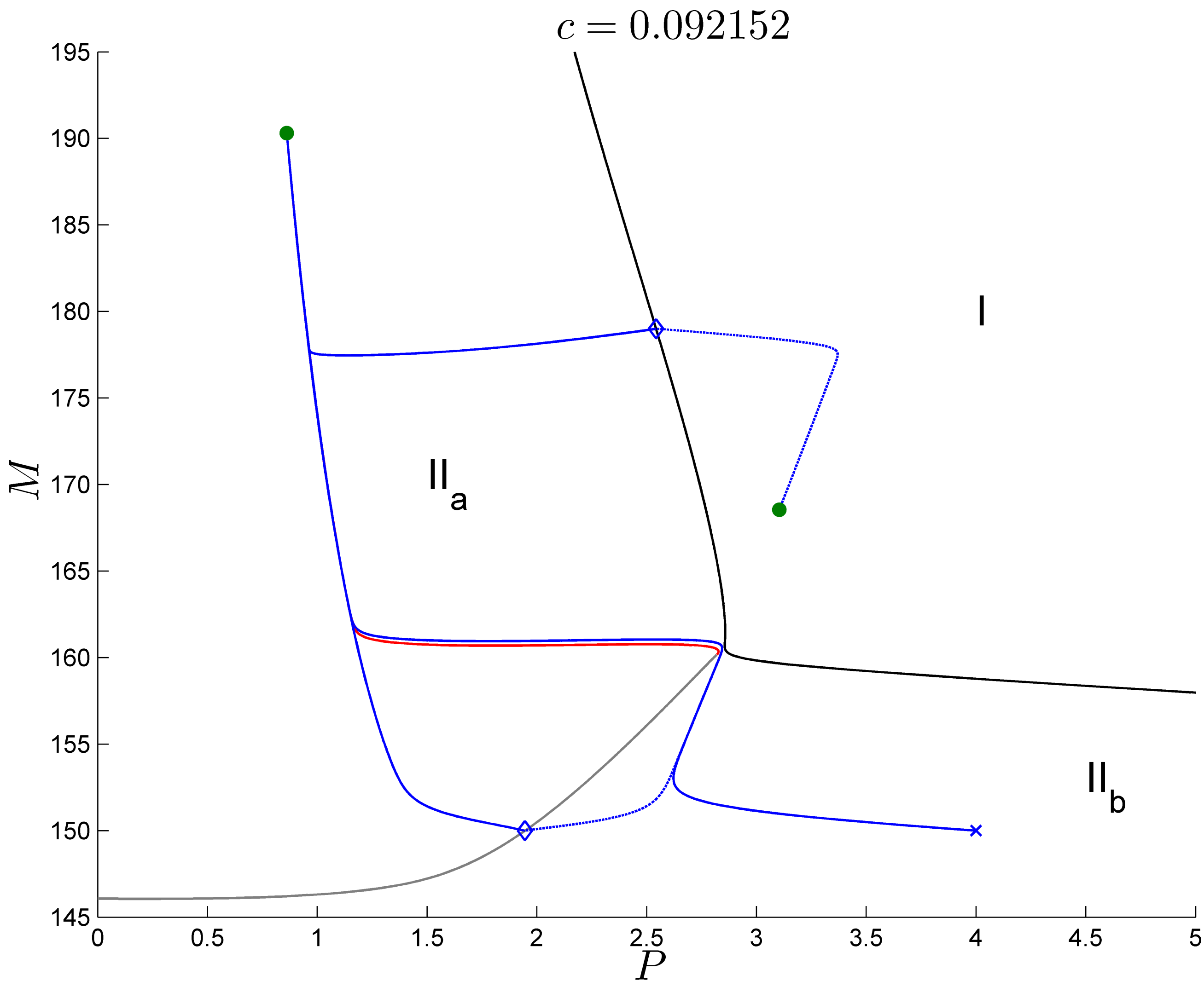

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