The Beijer Institute of Ecological Economics

DISCUSSION PAPER

Beijer Discussion Paper Series No. 148

Are social welfare functions ordinal or cardinal?

Karl-Göran Mäler. 2002.

The Beijer Institute of Ecological Economics
The Royal Swedish Academy of Sciences
Box 50005 SE-104 05 Stockholm Sweden
Phone: +46 8 673 95 00 Fax: +46 8 15 24 64
E-mail: beijer@beijer.kva.se

Are social welfare functions ordinal or cardinal?

Karl-Göran Mäler
The Beijer Institute
The Royal Swedish Academy of Sciences
Box 50005
104 05 Stockholm, Sweden
E-mail: Karl@beijer.kva.se
Phone: 46-(0)8-641 58 21
Fax: 46-(0)8-15 24 64

February 5, 2002

In a recent issue of this journal, Olof Johansson-Stenman (Johansson-Stenman 2000) had an interesting article on welfare wheights which gave me reason to revisit some earlier thoughts of mine (Mäler (1974) and Mäler (1985)).

I will discuss preferences over commodity bundles (x, Q), where x is a vector of net demand for marketed goods and services for which households are not rationed, while Q is a vector of goods and services which are rationed. Typically, we will interpret Q as a vector of environmental qualities. Furthermore, we will assume that Q is a vector of collective goods, that is all households face the same environment (we can always avoid this interpretation by introducing named goods).

Let there be H individuals, each with a preference order $\succsim_h$. From this preference order we define the strict order $\succ_h$ and indifference $\sim_h$. The problem is to aggregate these H preference orders to a social preference order $\succsim$:

$$\{\succsim_1, ..., \succsim_H\} \rightarrow \succsim$$

Assume that the aggregation satisfies the (weak) Pareto criterion:

$$\begin{aligned} If \quad (x^h, Q') &\succsim {}_h(y^h, Q'') \; \forall h, \text{ then } x \succsim y \; where \\ \text{x} &= ((\text{x}^1, Q'), ...(x^H, Q')) \text{ and y=((y}^1 Q'', ...(y^H, Q'')) \end{aligned}$$

Let u^h be a utility representation of $\succsim_h$, $h = 1, ..., H$. Such a utility representation is not unique and with any monotonically increasing function $\theta^h(.)$, $\theta^h(u^h(.))$ is a utility function representing the same preferences. A family of useful utility functions can be obtained in the following way[1] Let me assume that all goods are goods, that is more is always prefered to less of each good. This implies that there is no satiation. Let p be a strictly positive vector and define e^h from

$$e^h(p, Q, y) = \min\{px; (x, Q) \succsim (y, Q)\} \tag{1}$$

Given the vectors (p, Q), e^h will associate with each vector (y,Q) a real number which can be interpreted as the utility of the bundle y.

Before we come to the main argument in this note we need the concept of an indirect utility function.

Given the utility function $u^h(x^h, Q))$, the individual behaves *as if* she maximizes utility subject to the budget constraint $px \leqq I^h$, where p is the price vector and I the income of household h. This maximization give rise to the Marshallian demand functions

$$x^h = x^h(p, Q, I^h)$$

Substituting these functions into the utility function yields the indirect utility functions

$$v^h(p, Q, I^h) \equiv u^h(x^h(p, Q, I^h), Q)$$

Let the social welfare preference order be defined by a Samuelson-Bergson welfare function [2] W.

$$x \succsim y \text{ iff } W(u^1(x^1, Q'), ..., u^H(x^H, Q')) \geq W(u^1(y^1, Q"), ..., u^H(y^H.Q")) \tag{2}$$

As the individual preferences are ordinal and as the social preferences also are ordinal, it follows that a monotonically increasing transformation of individual utility functions should not change the social preferences[3]. A particular Samuelson-

[1] H. Uzawa was probably the first who used this method (Uzawa 1960).

[2] See Bergson (1938) and Samuelson (1947). The meaning of a Bergson-Samuelson welfare function is not always clear, see for example Sen (1986).

[3] Johansson-Stenman claims that "At least since Samuelson's classic text, it has been well-known that ...that consumer choices under ceretainty are independent of any positive monotonic transformation of the utility function. However, even at that time it was also clear that such transformations are generally not permissible in welfare analysis where some aggregated utility or welfare measure is used." Compare this with the following sentence from Samuelson's classical book p. 228: *But the W function is itself only ordinaly determinable so that there are an infinity of equally good indicators of it which can be used.* Even in Graaf's classical textbook (DeGraaf 1957), one can find similar remarks (pp. 35-38). Thus it is not at all clear from earlier writers that one needs cardinality as soon as we use a social welfare function! The major point with this note is that as long we limit ourselves to certain and à temporal analysis, we can build the whole analysis on ordinality.

Bergson welfare function is defined for a set of particular utility representations of the individual preferences. If we change these representations, we *must* change the corresponding welfare function in order not to change the social preferences. Thus, if we subject individual utility function u^h to the monotonically increasing transformation θ^h, W((.), ..., (.)) must change[4] into

$$W(\theta^{1^{-1}}(.), ..., \theta^{H^{-1}}(.))$$

where $\theta^{h^{-1}}$ is the inverse of θ^h. It is clear that with this welfare function and with the transformations θ^h, we will have the original social order, as

$$W(\theta^{1^{-1}}(\theta^1(u^1)), ..., \theta^{H^{-1}}(\theta^H(u^H))) \equiv W(u^1, ..., u^H)$$

Why are we interested in subjecting utility functions to monotonically increasing transformations? The answer is that some utility representations are easier to deal with than others. By making a transformation, we may end up with a particularly simple utility representation. The class of simplest utility representations is probably the class of representations such that utility can be measured in monetary terms. If we, in the construction of the utility function e^h above, let the vector p represent the market prices in the original situation and Q the supply of the arationed goods in the initial situation, the corresponding utility differences will represent the equivalent variation of the corresponding changes in the economic environment (and if we are comparing two situations and p and Q correspond to the second situation, the utility difference will correspond to the compensating variation).

By substituting the indirect utility function $v^h(.)$ for the direct utility function, we have an induced preference order on the price vector p and the income distribution $(I^1, ..., I^H)$ defined by

$$V(p, Q, I^1, ..., I^H) \equiv W(v^1(p.Q, I^1), ..., v^H(p, Q, I^H)) \tag{3}$$

It is routine to show that

$$\frac{\partial V}{\partial p_j} = -\sum_{h=1}^{H} x_j^h \frac{\partial V}{\partial I^h} \; and \; \frac{\partial V}{\partial Q_j} = \sum_h \frac{\partial V}{\partial I^h} \frac{\partial u^h}{\partial Q_j} \tag{4}$$

These equalities are intuitively obvious. The social planner would find social welfare unchanged if an increase in the price of a good with one unit is compensated

[4]This was observed by Paul Samuelson in Foundations of Economic Analysis (Samuelson 1947) and by de Graaf (DeGraaf 1957).

with income increases for all households with the cost of one unit of the same good. For a proof of 4 the reader is referred to Mäler (1974) .

It is interesting to note that if we start with the indirect social welfare function $V(p, I^1, ..., I^H)$ and impose on it condition 4 , one can show that this indiret welfare function can be derived from a direct welfare function (Mäler 1974). As it is easier in applied studies to start with the indirect social welfare function, we will discuss some of its properties.

First of all, note that V should be interpreted in an ordinal sense, that is, whenever we subject V to a monotonically increasing transformation, the new indirect social welfare function is as relevant as the original one. However, concavity is not a property that is invariant for monotone transformation, and concavity can therefore not be imposed on V as a function of incomes. Typically, concavity is imposed because it represents a desire for equity. Quasi concavity will, however, have the same implications. Thus, if the social preferences are such that, ceterus paribus, a more equitable distribution is preferred, then it make sense to assume V is quasiconcave in $I = (I^1, ..., I^H)$.This implies that the level sets $L = \{I^1, ..., I^H; V(I^1, ..., I^H, p) \geqq A\}$ for given pricevector p and for all A are convex and a lump sum redistribution of income such that the total income does not decrease and which would reduce income inequality will be an improvement of social welfare.

Consider now an initial situation A, characaterised by (p$'$, Q', I') and a final situation B, characterised by (p'', Q'', I''). What is the welfare change if the economy goes from A to B? The individual utility changes are given by

$$\Delta u^h = v^h(p'', Q'', I^{h\prime\prime}) - v^h(p', Q', I^{h\prime}) \tag{5}$$

By using the expenditure function, we can get an equivalent (but of course not an identical numerical) representation of this change:

$$\begin{aligned} \Delta e^{h\prime} &= e^h(p', Q', u^{h\prime\prime}) - e^h(p', Q', u^{h'}) = \\ &= e^h(p', Q', u^{h\prime\prime}) - I^{h\prime} = \\ &= e^h(p', Q', u^{h\prime\prime}) - I^{h\prime\prime} + (I^{h\prime\prime} - I^{h\prime}) = \\ &= e^h(p', Q', u^{h\prime\prime}) - e^h(p'', Q'', u^{h\prime\prime}) + \Delta I^h \end{aligned} \tag{6}$$

But this last equation is nothing but the equivalent variation for the change from A to B. Thus, this representation of the individual change in utility is simply the equivalent variation EV^h of the change.

Similarly, we can also represent the utility change as the compensating variation:

$$\begin{aligned}
\Delta e^{h\prime\prime} &= e^h(p'', Q'', u^{h\prime\prime}) - e^h(p'', Q'', u^{h'}) = \\
&= I^{h''} - e^h(p'', Q'', u^{h\prime}) = \\
&= I^{h\prime} - e^h(p'', Q'', u^{h\prime\prime}) + (I^{h\prime\prime} - I^{h\prime}) = \\
&= e^h(p', Q', u^{h\prime}) - e^h(p'', Q'', u^{h\prime\prime}) + \Delta I^h \qquad (7)
\end{aligned}$$

Obviously, this is the compensating variation (with conventional definitions of the compensating variation, this is the negative of CV^h.

Obviously, we could have started with any arbitrary feasible price vector p and and feasible supply of rationed goods Q, and m(p, Q, u) would be a valid utility representation of the underlying preferences.

In order to continue, let us decide to use the utility representation given by the equivalent variation (it would be extremely easy to instead use, say the compensating variation or any other representation based on the expenditure function). With these utility representations, the social welfare function will be denoted W_{EV}. The change in social well-being can now be written

$$\begin{aligned}
\Delta W_{EV} &= W''_{EV} - W'_{EV} = \\
&= W_{EV}(e^1(p', Q', u^{1\prime\prime}), ..., e^H(p', Q', u^{H\prime\prime})) - \qquad (8) \\
&\quad -W_{EV}(e^1(p', Q', u^{1\prime}), ..., e^H(p', Q', u^{H\prime})) \qquad (9)
\end{aligned}$$

This can be approximated by a linear expression if the change from A to B is small:

$$\Delta W_{EV} \approx \sum_{h=1}^{H} \frac{\partial W_{EV}}{\partial e^h} EV^h \qquad (10)$$

The weights, $\frac{\partial W}{\partial e^h}$, are of course depending on the utility representation, and a change in this representation will of course change these weights! For example, with the CV representation, the wheigts will be different, as they should in order to guarantee that the welfare change should have the same sign as it has when we are using the EV representation.

Assume that the socialwelfare function is quasiconcave for one utility representation, it easy to see that it must be quasiconcave for all such utility representations of individual preferences. The degree of quasiconcavity is a measure of the social preferences for a more even "utility" distribution. In the case society has no preferences for the distribution of utilities (as represented say by the EV, the weights will be constant and equal for all individuals. However, for another utility representation, the weights will be different and not necessarily equal across the population. The reason is of course that when we go from say the CV representation to the EV representation, we are subjecting the utilities to a non-linear monotonic transformation. If we insist that the welfare weights must be equal, we are forced to reject the idea of an ordinal representation and instead we have to work with a cardinal one. in which only linear transformations are acceptable. But there is no reason why we should insist on that. And if we do not insist, then we just have to take note that the welfare weights will vary with the utility representations, and as long as we do not try to mix different utility representations, everything will work out correctly!

References

Bergson, A. (1938), 'A reformulation of certain aspects of welfare economics', *Quarterly Journal of Economics* **52**(1), 310–334.

DeGraaf, J. V. (1957), *Theoretical Welfare Economics*, Cambridge University Press, Cambridge, UK.

Johansson-Stenman, O. (2000), 'On the value of life in rich and poor countries and distributional weights beyond utilitarianism', *Environmental and Resource Economics* **17**(3), 299–310.

Mäler, K. (1974), *Environmental Economics: A Theoretical Inquiry*, John Hopkins University Press, Baltimore.

Mäler, K. (1985), *Welfare economics and the environment, in: A.V. Kneese and J.L. Sweeney eds., Handbook of Natural Resource and Energy Economics, Vol. 1*, North Holland, Amsterdam.

Samuelson, P. A. (1947), *Foundations of Economic Analysis*, Harvard University Press, Cambridge, Ma.

Sen, A. (1986), *Social Choice Theory*, Vol. III of *Handbook of Mathematical Economics*, North Holland, chapter 22.

Uzawa, H. (1960), Preference and rational choice in the theory of consumption, *in* S. K. Kenneth J. Arrow & P. Suppes, eds, 'Mathematical Methods in the Social Sciences, 1959', Stanford University Press, pp. 129–148.