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# An Ecological Golden Rule

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# An Ecological Golden Rule*

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**Abstract**

Most renewable biotic resources are subject to random variability in natural growth. We investigate the implications of such variability for long-term management by a risk averse social planner who maximizes expected long-run utility. In a simple model of a stochastic fishery, we show that the optimal level of harvesting effort need not necessarily be reduced by the introduction of variability in stock growth. However, optimal effort is reduced if the variability of growth increases for smaller base populations, as suggested in the ecology literature.

**JEL classification: D90, D63, Q20, Q22, C62**

**Keywords: Stochastic Growth, Natural Resource, Golden Rule**

## 1 Introduction

The natural growth of many biotic resources and ecological populations is subject to substantial random fluctuations, either as a result of natural variation or stochastic environmental conditions. Natural variability in stock levels can translate into variability in harvests, particularly if harvesting effort levels are not adjusted rapidly enough to adapt to short term fluctuations, potentially leading to loss of welfare.

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It is sometimes suggested in the Ecology literature that stochastic variability in a species' growth implies the level of harvesting effort should be reduced (for example, Roughgarden and Smith (1996)). Our analysis examines the question in a simple model of a stochastic fishery, in which the welfare costs of variable harvests are captured by a concave utility function. Our main result in this setting is that variability in stock growth need not necessarily reduce the optimal level of effort. We show that the crucial characteristic of resource growth is not the degree of variability, but whether this variability is higher at lower stock levels. It is only under this condition that the optimal level of effort is reduced in comparison with the deterministic model.

We note that increasing variability of growth at lower stock levels is a plausible attribute of real ecological systems (see, for instance, Lande (1993)). One reason for demographic variability to increases at low population levels is that the group average of the independent chances of individual mortality and reproduction growth rates fluctuates around the mean, with a variance that scales inversely with the size of the population. Another could be that smaller populations are more vulnerable to the impacts of environmental conditions (Leizarowitz and Tsur (2012)).

The Economics literature on the management of stochastic natural resources has focused on the implication of stochastic growth for the nature of the inter-temporally optimal policy, i.e. the harvesting policy that maximises the present discounted utility of harvest, and the possibility of extinction (Reed, 1975, 1978, 1979; Beddington and May, 1977; Getz et al., 1987; Costello et al., 2001; Sethi et al., 2005; Mitra and Roy, 2006; McGough et al., 2009; Leizarowitz and Tsur, 2012). This paper departs from this framework in that it is concerned with a long-term perspective, and thus studies a simpler welfare criterion focused on the long-run level of the expected utility of harvest. The problem is a natural extension of the *Golden Rule* of saving (Phelps, 1961) and the more recently introduced *Green Golden Rule* of resource extraction (Chichilnisky et al., 1995) to a stochastic framework. In this context, we examine the impact of random variability in stock levels for choice of the optimal level of harvesting effort. In our model, there is no possibility of extinction, but steady state variability in catch levels is welfare reducing.

The paper is organized as follows. In section 2, the general resource extraction problem is set up, with the deterministic version discussed in section 3. Section 4 considers this problem under stock-dependent uncertainty, and establishes a result analogous to the 'Green Golden Rule' of Chichilnisky et al. (1995) for effort level. Section 5 concludes. All technical details are collected in the Appendices.

## 2 Resource Dynamics

To illustrate our ideas, we introduce stochastic growth factors into the canonical model of fishery dynamics used in the seminal work of Levhari and Mirman (1980). The model is also formally equivalent to a Solow saving model with a Coub-Douglass production function. Let the size of the fish population at time t be given by $X_t$. The natural dynamics of the fishery is given by the equation

$$X_{t+1} = g_t X_t^{\alpha}, \tag{1}$$

where $0 < \alpha < 1$ and $g_t$ are stochastic (random variables) growth factors, whose distribution $F(.; X)$ can depend upon the stock level $X$, but which we otherwise assume to be stationary.

The social planner chooses a level of harvest effort, $e \in [0, 1]$, which describes the number of boats, the time spent fishing, etc.. The level of effort embodies a choice that is hard to change over time and is represented in the model by a constant.[1] We follow standard models by assuming that the catch (fish consumption) is proportional to both the (regenerated or 'grown') stock level and the level of effort, i.e.

$$C_{t+1} = e g_t X_t^{\alpha} \tag{2a}$$

$$X_{t+1} = (1 - e) g_t X_t^{\alpha}. \tag{2b}$$

[1] The assumption of constant effort is quite common (see e.g. Beddington and May (1977) and references therein) and can be justified as being not far from optimal (Parma (1990)).

Finally, the utility of consumption is assumed to be logarithmic,

$$U_t = \log(C_t). \tag{3}$$

The variables $g_t$, $X_t$, $C_t$ and $U_t$ are random variables, and eq. (2b) describes a Markov process. A formal discussion of the convergence properties of this system are to be found in the Appendix of the working paper version , but we note here that under common assumptions made in discrete-time stochastic dynamic models, the system converges in probability to a steady state represented by random variables $X^\star$, $C^\star$ and $U^\star$.

We define the ecological golden rule as the level of effort, $e$, which maximizes asymptotic expected utility,

$$\widehat{e} := \underset{e\in[0,1]}{\arg\max} \left\{\mathbb{E}\left(U^\star(e)\right)\right\}. \tag{4}$$

## 3 The Deterministic Case

We begin by revisiting the familiar deterministic case and then introduce stochastic growth. In the deterministic case, $g_t \equiv 1$, so that natural growth is described by

$$X_{t+1} = X_t^\alpha. \tag{5}$$

Steady states stock levels are given by $X = 0$ (unstable) and $X = 1$ (stable). With harvesting effort e, the system dynamics are simply

$$C_{t+1} = eX_t^\alpha, \tag{6}$$

$$X_{t+1} = (1-e)X_t^\alpha,$$

$$U_{t+1} = \log(C_{t+1}) = \log(e) + \alpha\log(X_t). \tag{7}$$

In this model, every effort level stabilizes the system in a finite steady state, with the following stock and utility:

$$X^\star = (1-e)^{\frac{1}{1-\alpha}} \tag{8a}$$

$$U^\star = \log(e) + \frac{\alpha}{1-\alpha}\log(1-e). \tag{8b}$$

The *golden rule* level of effort strives to balance the positive effect of effort on harvest amounts with its the negative effect on stock levels. By differentiating with respect to $e$, it is easy to check that this is given by

$$e^\star = 1-\alpha. \tag{9}$$

Two simple points regarding this set up are worth mentioning. First, in the deterministic case, maximizing the steady state level of utility of consumption is equivalent to maximising the steady state level of consumption itself, so the utility function plays no role in this model. This will no longer be true in the stochastic case. Second, we have not introduced a cost to exerting effort into the model. Nonetheless, even when effort is costless, it is not optimal to set effort to its maximal level, $e = 1$. Increasing effort increases consumption *given the level of the stock*, but it also reduces the level of the stock in steady state. The balance between these two effects is struck by the golden rule, and is completely analogous to the golden rule level of saving in the Solow growth model.

# 4 The Stochastic Fishery

We now incorporate stochastic growth and return to the full model described by eq. (2).

As already discussed, the crucial property of the growth distribution, as is often suggested in the ecology literature, is that growth factors are more likely to be lower when stock levels are lower, i.e. that $F(g;X) \leq F(g;X')$ whenever $X \geq X'$. For tractability, we impose a specific form on this distribution that allows us to capture the main insight transparently.[2] We will assume that $g_t$ is log-normally distributed

$$g_t|X_t \sim \log\mathcal{N}(\mu(X_t), \sigma^2(X_t)),$$

and impose the specific functional form for its moments:

$$\mathbb{E}(g_t|X_t) = 1, \tag{10a}$$

$$\mathbb{V}(g_t|X_t) = vX_t^{-2\lambda} - 1. \tag{10b}$$

In the first equation, we assume, for convenience and without loss of generality, that the growth factors all average to 1. In the second equation, we impose a specific form on the variance of the growth distribution. Here, $v - 1$ is the variance of $g_t$ when $X_t = 1$ (we assume that $v \geq 1$),[3] and $\lambda \geq 0$ is a parameter that describes how strongly variability in growth rates declines with stock levels.When $\lambda = 0$, the variance (and the entire distribution of $g$) is independent of the level of the stock. The larger the value of $\lambda$ is, the more rapid is the increase in variance at lower levels of the stock. To satisfy the moments detailed in eq. (10), the scale of the log normal distribution will need to be set to $\sigma^2(X_t) = \frac{1}{2}\log(v) - \lambda\log(X_t)$.[4]

The steady state of stochastic dynamical system[5] represented by eq. (2b) is itself a probability distribution over stock levels (inducing a distribution over consumption and utility). To find the form of the steady state distribution, we take the logarithms

[2] Appendix B details an alternative way of obtaining our key insights, directly working with the more general conditions on $F(.,X)$, for both the log-normal and the Weibull distribution.

[3] Clearly, this expression is sensible only when $X_t > 0$. The details regarding this aspect, as well as why it is always valid for the model we consider, are available upon request. Nonetheless, it is important to note that for the more general way of obtaining our key insights (detailed in Appendix B.1), we do not use the specific functional form in eq. (10b), as a result of which details regarding its parameterization do not in any way affect our main results.

[4] We recall a few basic properties of the log normal distribution: if $Z \sim \log\mathcal{N}(\mu, \sigma^2)$, then $\mathbb{E}(Z) = e^{\mu+\sigma^2}$ and $\mathbb{V}(Z) = \mathbb{E}(Z)(e^{\sigma^2} - 1)$. These expressions for the mean and variance, together with the moments required from eq. (10), will determine the scale, $\sigma^2(X_t)$, and location, $\mu(X_t)$, of the log normal distribution.

[5] The "steady state" of a stochastic dynamic system involves convergence of the distribution of the Markov chain that the stock evolution in eq. (2b) represents i.e. of $\Psi_t \to \Psi^\star$, with $\Psi_t$ the time $t$ measure and $\Psi^\star$ the so-called *invariant* measure (see Appendix C for details regarding the existence and uniqueness of the invariant measure). In the interests of keeping the exposition clear, we do not stress this aspect, subsuming this instead into the more common notion of $X_t \to X^\star$.

of both sides (of eq. (2b)) to obtain

$$\log(X_{t+1}) = \log(1-e) + \log(g_t) + \alpha \log(X_t).$$

Taking expectations,[6]

$$\begin{aligned}\mathbb{E}\left(\log(g_t)\right) &= \mathbb{E}_{X_t}\Big[\mathbb{E}\big(\log(g_t)|X_t\big)\Big] \\ &= \mathbb{E}_{X_t}\left(-\frac{1}{2}\log(v) + \lambda \log(X_t)\right) \\ &= -\frac{1}{2}\log(v) + \lambda \mathbb{E}\log(X_t),\end{aligned}$$

we find that

$$\begin{aligned}\mathbb{E}\left(\log(X_{t+1})\right) &= \log(1-e) + \mathbb{E}(\log(g_t)) + \alpha\mathbb{E}(\log(X_t)) \\ &= \log(1-e) - \frac{1}{2}\log(v) + (\lambda+\alpha)\,\mathbb{E}(\log(X_t)). \end{aligned} \tag{11}$$

The series of numbers $\mathbb{E}\left(\log(X_t)\right)$ converges to the steady level of expected stock level $\mathbb{E}\left(\log(X^\star)\right)$. Taking the limit of both sides of eq. (11), and then taking the expectation of the utility function, we find that in the steady state,[7]

$$\mathbb{E}\left(\log(X^\star)\right) = \frac{\log(1-e) - \frac{1}{2}\log(v)}{1-(\alpha+\lambda)}, \tag{12a}$$

$$\mathbb{E}\left(U^\star\right) = \mathbb{E}\left(\log\left(C^\star\right)\right) = \log(e) + \frac{\alpha+\lambda}{1-(\alpha+\lambda)}\log(1-e) - \left\{\frac{\log(v)}{2}\left[\frac{1}{1-(\alpha+\lambda)}\right]\right\}. \tag{12b}$$

In this set-up, the *Ecological Golden Rule* level of effort is the value of $e$ which maximizes eq. (12b), the expression for long-run utility. This level of effort needs to

[6]In the interests of emphasizing intuition, and minimizing notation, we will not be excessively formal as regards the meaning of "expectation" in the text. Thus, all expectations are to be understood to be w.r.t. either $\Psi_t$ or the invariant measure, $\Psi^\star$, as appropriate.

[7]An important point to note is that we assume, in eq. (12a), convergence of not just the Markov Chain, $\{X_t\}$, but also of moments of a functional of this chain, $\log(X_t)$. It is not generally true that convergence of a Markov process process implies convergence of the moments of the process, much less of a functional of the process. Arguments sketching out a proof of this fact are presented in Appendix D.

balance, as in the deterministic case, the benefits of higher consumption (for a given the stock level) with the steady state (expected) level of the stock; in addition, it also must consider the steady state variance in the level of the stock. The latter is important because it is directly related to the variance of (steady state) consumption, which, because of the concavity of the utility function, reduces the expected utility of consumption.

An examination of eq. (12b) leads to several observations.[8] In the simplest case, there is no variance in the growth rate ($v = 1$, $\lambda = 0$) and we naturally recover the deterministic result. If, however, there is uncertainty in growth rate, $g_t$, which is independent of the level of the stock (i.e. $\lambda = 0$ but $v > 1$), then eq. (12b) differs from its deterministic version in eq. (8) by an additive, negative constant independent of $e$. This latter feature has an important and intuitive implication, that for any given level of effort, utility with randomness in growth is lower than in the deterministic level (recall that $v \geq 1$).[9] This is just an expression of the risk aversion embodied in the utility function. More importantly, since this utility reduction is independent of the level of effort, it also means that the optimal level of effort is unaffected by the random nature of the growth factors.

In the more interesting case, and the one we focus on, with stock-dependent variance (i.e. $\lambda > 0$ and $v > 1$), a comparison of eq. (12b) and eq. (8) yields an additional (to the downward shift in utility observed also for the case of stock-independent variance) effect, which is that $\lambda$ exerts an identical effect on optimal effort as does natural growth, $\alpha$. In particular, we find that the optimal level of effort is shifted downward by $\lambda$,

$$e^{\star} = 1 - \alpha - \lambda. \tag{13}$$

In other words, the golden rule level of effort is lower when there is stochastic growth

[8] Clearly, the results in eq. (12b) only make sense for $\alpha + \lambda < 1$. Overall, then, the results here pertain to a system in which growth of stock is only moderate (low $\alpha$), and so too is the effect of stock upon variance i.e. $\lambda$ is 'small'.

[9] Comparing the expression for $U^{\star}(C)$ from eq. (8b) and $\mathbb{E}(U(C))$ from eq. (12b), we see that $U^{\star}(C) - \mathbb{E}_{\Psi}(U(C)) = -\dfrac{\lambda \ln(1-e)}{(1-\alpha)(1-\alpha-\lambda)} + \dfrac{\ln(v)}{2}\left[\dfrac{1}{(1-\alpha-\lambda)}\right] \geq 0.$

with stock-dependent variance, unless $\lambda = 0$. When $\lambda = 0$, the variability in growth rates is independent of the level of the stock, so it makes sense for the choice of the optimal effort to be unaffected. In all other cases, though, the higher variability in growth for lower levels of the stock makes it desirable to maintain the stock at a higher level. Our main result can be summarized as:

**Proposition 1.** *In the model of resource extraction represented in eq.* (2) *and eq.* (4)*, with growth factors log normally distributed as in eq.* (10)*, the optimal ('golden rule') level of effort is reduced as a result of uncertainty in stock if, and only if, the variance in growth rates is larger at lower stock levels. More generally, for the model in eq.* (2) *and eq.* (4)*, the optimal level of effort is lower with uncertainty in stock growth whenever higher growth rates are more likely under higher stock levels.*

### 4.1 Discussion

We note that our problem bears a resemblance, albeit faint, to the deterministic problem in Chichilnisky et al. (1995), whose objective is to find feasible optimal paths for long run utility (when utility is a function of natural stock and man-made capital). Theirs is a problem closely related to that of economic growth, leading to their calling the optimal consumption path for their problem the 'Green Golden Rule'. Our problem is different in fundamental ways, not least of which is the constant-over-time effort levels. Nonetheless, our optimal effort levels are always feasible and sustainable, our optimality criteria, which can be formally written out as $\max_{e \in (0,1)} \lim_{t \to \infty} \mathbb{E}\Big[U\left(C(X_t, e)\right)\Big]$, bears a resemblance to theirs, and both eschew the discounted utilitarian objective. Given these similarities, and the ecologically-based motivation for our stock dynamics, we term our optimal effort levels the 'Ecological Golden Rule' level of effort.

Our main result, proposition 1, turns out to have an intuitive and interesting economic interpretation. Recall that effort levels–being time invariant– parameterize the (time $t$ and long-run) stock distribution, and that, it turns out, in a particularly simple way: every level of effort may be viewed as leading to a lottery over (long-run) consumption. In this interpretation, the main question is one related to choice of the optimal lottery. Indeed, it follows that there is a specific form of the mean-variance trade-off (over consumption) involved in this choice: increased effort directly increases the mean but also, equally directly, the variance. This direct increase in variance is an additional source of disutility to the risk averse decision maker.

## 5 Conclusions

This paper is concerned with analyzing the implications, for optimal management of a renewable resource, of a relationship long held as important by ecologists: that stock-dependent uncertainty in growth rates are important for understanding population dynamics. A simple but rigorous stochastic dynamic bio-economic model is employed, with consumption of the renewable resource providing utility while depleting the stock of the resource, which is replenished with a random growth factor which varies with stock level. In contrast to many existing studies that are focused on maximizing discounted utility, we are concerned with maximizing long-run utility, emphasizing the connection with sustainability.

Our main finding is that extraction effort is *lower* in cases where the variance of natural growth rate of stock is higher for lower stock levels, an important feature of many ecological systems. This is, to our knowledge, the first formal analysis of economic management of ecological systems with such a feature. An important contribution of this paper lies in identifying a new mechanism, one ecologically based, governing harvest effort in dynamic renewable resource systems. Previous mechanisms have been focused on substitution, between man-made and natural capital (in a deterministic framework, in Chichilnisky et al. (1995)) and between bio-diversity and consumption (in a stochastic framework, Li et al. (2001)). What we identify is an entirely new mechanism: an intrinsic link between stock level and its variability.

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# Appendix A Basic Setup

We gather here a few miscellaneous details regarding a few features of the model discussed in section 4 and discuss a few details of the specification of the variance (from eq. (10b)). Our basic set-up closely follows the large literature in economics dealing with stochastic dynamic resource models. Stock, $X_t$, is considered to be drawn from a compact set, $\mathcal{X} \subset \mathbb{R}_+$, as a result of which it is possible to normalize $X_t$ to lie within the unit interval (excluding zero).

Thus, both in the deterministic model (as evident from eq. (8b)) and the stochastic model (as will be justified next), the steady state (and for each $t$) stock is seen to lie in the unit interval. Continuing with the assumption that $X_t \in (0, 1]$, $\forall t$ (for now), we next note that the combination of parameter values for natural growth ($\alpha$) and the degree of variance dependence ($\lambda$) satisfy the condition $\alpha + \lambda < 1$. In other words, only small-to-moderate values of these parameters are admissible. All these considerations ensure that the condition for positive variance (from eq. (10b)), $v > X^{2\lambda}$, is easily met.[10] To summarize, choosing $v = 1 + \epsilon(\lambda)$ will ensure that $v > X_t^{2\lambda}$, $\forall t$, so long as at least $X_t \in (0, 1]$, $\forall t$.

It is worth emphasizing that these problems involving careful choice of $v$ and $\lambda$ turn out not to matter when the more general probabilistic representation (detailed in Appendix B.1 ) is used to derive the expression for $\mathbb{E}(\log g_t)$. Thus, our main insights do not particularly depend upon the specific choice of functional form for stock-dependent variance made in eq. (10b) (the parameter choices for which motivated the discussion above).

[10]To see this explicitly, consider two levels for stock, $X_t$, $\frac{1}{4}$, $\frac{3}{4}$, and the same two values for $\lambda$, $\frac{1}{4}$, $\frac{3}{4}$; these represent 'low' and 'high' (resp.) stock levels and parameter values (noting that $\lambda < 1$). In all of these cases, $v < 1$ will suffice; indeed, this is true more generally so long as $X$ is uniformly bounded away from 0 (i.e. $X_t > M > 0$, $\forall t$), a standard assumption in many dynamic stochastic economic models of renewable resources .

# Appendix B Other Distributions

We indicate, following Stachurski (2003, p. 144), a general way in which one may generate a random variable from a distribution function which satisfies the required condition, $F(g;X) \leq F\left(g;X^{'}\right)$, for $X > X^{'}$. Let $\psi$ denote the density function for the shock (growth factors, in our case), and set

$$F_X(g) = \int_0^{\frac{g}{\nu(x)}} \psi(s)ds, \tag{B.1}$$

where $\nu(X)$ is an increasing function of $X$ (for instance, set $\nu(X) = aX + b$ for $a, b > 0$, or $X^a$ for $a > 0$). It can easily be verified that this formulation above satisfies the required condition.Using this method, we prove that proposition 1 holds whenever $g$ has either the log normal or the weibull distribution.

## B.1 Lognormal distribution

Let $\psi$ represent the density function of a log normal random variable, with parameters $(\mu, \sigma^2)$. Consider the integral below, which is the distribution function of $g_t$:

$$F_X(g) = \int_0^{g/\nu(X)} \psi(s)ds = \Phi\left[\frac{\log\left(\frac{g}{\nu(X)}\right) - \mu}{\sigma}\right] = \Phi\left[\frac{\log(g) - \tilde{\mu}(X)}{\sigma}\right].$$

$\Phi$ is the distribution function of a standard normal variate and $\tilde{\mu}(X) = \mu + \log(\nu(X))$. Thus, we have that $g_t \sim \mathcal{LN}\left(\tilde{\mu}(X), \sigma^2\right)$. Recalling the fact that if $g_t \sim \mathcal{LN}(\tilde{\mu}(X), \sigma^2)$, $\log(g_t) \sim N(\tilde{\mu}(X), \sigma^2)$, we have that $\mathbb{E}\left(\log g_t\right) = \tilde{\mu}(X) = \log(\nu(X)) + \mu$. Noting that the preceding analysis (implicitly) conditions on $X$, we see that one can use the same 'tower property' for conditional expectation (as used in section 4) leading to results identical to that in section 4. More explicitly, setting $\nu(X) = X^a$, for $a > 0$ we again have that

$$\mathbb{E}(\log g_t) = \mathbb{E}\left(\mathbb{E}_X(\log g_t | X_T)\right) = \mathbb{E}_X\left(\tilde{\mu}(X)\right) = a\mathbb{E}(\log X_t) + \mu. \tag{B.2}$$

We note that here we have made no special assumptions on the parameter values for $\mu$ and $\sigma$. It is easy to check that using this expression for $\mathbb{E}(\log g_t)$, eq. (12b) is unchanged (after identifying $\mu$ here with $-\frac{1}{2}\log(v)$, and $a$ with $\lambda$), and so is our main result, proposition 1 . As an added benefit, it should be evident that this way of deriving the expression for $\mathbb{E}(\log g_t)$ also obviates the care necessary to ensure that the parameterization of the stock-dependent variance specification, in eq. (10b), is sensible.

### B.2 Weibull distribution

Let now $\psi$ be the density function of the two-parameter weibull distribution, which implies that $\psi(s;\beta,\theta) = \frac{\beta}{\theta}\left[\frac{s}{\theta}\right]^{\beta-1}\exp\left\{-\left(\frac{x}{\theta}\right)^{\beta}\right\}$ where $\theta,\beta > 0$ and, w.l.o.g, we set $\theta = 1$. Following an identical procedure as for the log normal case, we have that

$$F_X(g) = \int_0^{g/\nu(X)} \psi(s)ds = 1 - \exp\left(\frac{-g}{\nu(X)}\right)^{\beta},$$

which implies that $g_t \sim \text{Weibull}\left(\beta,\nu(X)\right)$. For $T \sim \text{Weibull}(\beta,\nu(X))$, it is the case (see e.g. White (1969, p.375)) that $\log T \sim$ log-Weibull, with the distribution function

$$F_X(t) = 1 - \exp\left(-\exp\left[\beta(t - \log\nu(X))\right]\right).$$

More importantly for the present purpose, its moments maybe computed through a simple transformation[11] (see White (1969, pp 375-76) and also Johnson et al. (1994, Vol. 1, pp 635 and Vol 2. pp 2-3)) yielding

$$\mathbb{E}(\log g_t) = \frac{-\gamma}{\beta} + \log(\nu(X)) = \frac{-\gamma}{\beta} + a\mathbb{E}(\log X). \tag{B.3}$$

Thus, $\mathbb{E}\left(\log g_t\right)$ is again of the same form as in section 4, and the remarks following eq. (B.2) are directly applicable.

[11] $\gamma$ is the EulerMascheroni constant obtained here as a result of differentiantion of the $\Gamma$ function i.e. $\Gamma'(1) = -\gamma$

# Appendix C Existence of a Unique, Stationary Distribution

In a dynamic stochastic setting, it is of substantial importance to understand when the sequence of distribution functions, denoted $\{\phi_t\}$, associated with the evolving random variables, $\{X_t\}$, converge to a unique distribution, $\phi$, called the steady-state distribution. In general, apart from when the evolution of $X_t$ are linear and Markovian, the existence of such distributions are not guaranteed. In the special case considered here, the existence of such an invariant distribution follows from results in (Stachurski, 2003), which we elaborate on here. We note that when the shocks are not i.i.d and in fact depend on the state, there are few general approaches or results which are directly applicable. An important reason for presenting a proof here is that standard methods for proving global stability in economics, e.g. (Hopenhayn and Prescott, 1992) and (Stokey and Lucas, 1989), are not applicable for this case.

We prove rigorously here that the following Markov chain

$$X_{t+1} = (1-e)g_t X_t^{\alpha},$$

with $g_t \geq 0$ a state-dependent random noise, $X_t \in \mathcal{X} \subset \{\mathbb{R}_+ \setminus \{0\}\}$, $e \in (0,1]$, $1 > \alpha > 0$, has a unique invariant distribution. We shall not be overly formal in the presentation and will instead focus on sketching out the important steps involved. We begin by first collecting some terminology, assumptions and conditions. Let $f(x) = (1-e)X_t^{\alpha}$, implying

$$X_{t+1} = g_t f(X_t), \tag{C.1}$$

where as in Appendix B, $g_t$ is such that $F(g;X) \leq F(g;X')$ whenever $X \leq X'$. We note first that the stochastic recursive system in eq. (C.1) corresponds identically to the case of "growth with state-dependent" shocks in Stachurski (2003, 4.2), to which we direct the readers for fuller details. Consequently, we will only outline how, in our case, all the conditions required for the existence of a unique, stationary distribution are met.

The model in eq. (C.1) can actually be recast into a model with i.i.d errors as follows: consider a random variable $\eta_t$ with density $\psi$; then, from the notation and definitions in Appendix B, it is not hard to see that $g_t \stackrel{d}{=} \eta_t \nu(X)$ i.e. $g_t$ has the same distribution as $\eta_t \nu(X)$.Using this observation, the SRS in eq. (C.1) can be rewritten as

$$X_{t+1} = \eta_t \underbrace{\nu(X_t) f(X_t)}_{\hat{f}} = \eta_t \hat{f}\left(X_t\right). \tag{C.2}$$

We state next two general assumptions that hold for reasonable distributions with density, $\psi$, and for a function $\hat{f}$.

**Assumption 1** $\hat{f} > 0$ *a.e.* on $\mathbb{R}_+$.

**Assumption 2** Shocks $\eta_t$ are i.i.d, with a density $\psi$.

**Condition 1** Corresponding to $(\hat{f}, \psi)$, there exists a Lyapunov function $V$ on $\mathbb{R}_+$, $\delta < 1$ and $\infty > C > 0$ s.t.

$$\int V\left[\hat{f}(x)z\right]\psi(z)dz \leq \delta V(x) + C, \forall x \in \mathbb{R}_+.$$

**Condition 2** $\psi > 0$ a.e.

**Condition 3** For some $M < \infty$, $\psi(z)z \leq M$, $\forall z \in \mathbb{R}_+$.

Note that Assumption 1 and 2 are true for our model for most common distributions for $\eta$, and so is Condition 2. We state an additional Condition before stating the main result of this section and subsequently indicate why Condition 1, and an additional condition required for the result below, are met by our model.

**Condition 4** The Markov Chain, $\{X_t; t = 1, 2, \ldots\}$, is defined on a compact space i.e. $\mathcal{X}$ is a compact subset of $\{\mathbb{R}_+ \setminus \{0\}\}$.

**Theorem C.1.** *Let $(\hat{f}, \psi)$ satisfy Assumption 1 and 2 and Conditions 1-4. Then, $(\hat{f}, \psi)$ has a unique, globally stable equilibrium.*

*Proof.* This is Proposition 7 in Stachurski (2003, §3) □

Note that Proposition 7 in Stachurski (2003, §3) requires that, for some constants $0 < \nu' < \nu'' < \infty$, it is the case that $\nu' \leq \nu \leq \nu''$. It is evident that this condition is satisfied whenever $\nu$ is continuous and $\mathcal{X}$ is compact. Thus, Condition 4 ensures that, for any reasonable function $\nu$–as for the power function used here–this requirement is met.[12]

It only now remains to verify that Condition 1 holds i.e. to find a function $V$ satisfying the specified conditions. Given that our function $\hat{f}$ is identical (to multiplicative constants) to that in Stachurski (2003, 4.2), the function used there, $|\log(x)|$, will suffice for our case too, as can be verified by an inspection of the proof of Proposition 7.

# Appendix D Convergence of Moments

An important point to note is that we assume, in eq. (12a), eq. (12b), convergence of not just the Markov Chain, $\{X_t\}$ but also of moments of this chain. It is not true in general that convergence of the process implies convergence of the moments of the process. The issues involved are detailed here, again at a relatively informal level. Denote by $\pi$ the stationary measure, and by $\pi_t$ the measure of the Markov process at time $t$, where we use different notation from the previous sections for consistency with key references.

Observe that we assume above, when using "steady state" results in eq. (12b) and in Proposition 1, that $\lim_{t\to\infty} \mathbb{E}(\log(X_t)) := \lim_{t\to\infty} \int_S \log(x)\pi_t(dx) = \int_S \log(x)\pi(dx) < \infty$. In other words, we have assumed that not only do the moments of $X_t$ converege, i.e. that $\lim_{t\to\infty} \mathbb{E}(X_t) = \int_S x\pi(dx)$, but that moments of a *functional* of $X_t$, $\log(.)$ in this case, converge. We will lead up to our main result by means of a series of remarks and propositions, making use of the model primitives already in force.

[12] We observe that compact state spaces are a virtual necessity for stochastic dynamic programming and are virtually always assumed in economic models. Further, we note that the compactness assumption may actually be weakened, at the cost of only a slightly more involved argument; the key requirement is actually that the state space not include 0 stock, a condition that has also been used in related studies, e.g. (Olson and Roy, 2000).

**Remark D.1.** *We will continue to use Condition 4 in Appendix C, implying that our function of interest,* log *will be bounded and continuous on* $\mathcal{X}$*. Assumption 2 and Condition 2, implying that there is a strictly positive density* $\psi$ *will also be in force. This, together with the continuity of the function* $\hat{f}$ *ensures that the transition probability function* $P$ *of the Markov chain* $\{X_t; t = 1, 2, \ldots\}$ *is "strongly Feller" i.e. that the transition probability has a density . Finally, from the assumptions so far made, it is also evident that* $\log(x)$ *can be assumed non-negative w.l.o.g.*

The conditions of Remark D.1 lead to the following result that essentially guarantees that the Markov chain $\{X_t; t = 1, 2, \ldots\}$ is irreducible. Note that our proof of global stability, Theorem C.1, did not require irreducibility of the Markov chain.

**Proposition 2.** *Under the conditions in Remark D.1, the Markov chain* $\{X_t; t = 1, 2, \ldots\}$ *is irreducible.*

*Proof.* This is a direct consequence of Corollary 7.2 in Tuominen and Tweedie (1979, 7). See also Tuominen and Tweedie (1979, p.89) □

Theorem 3 and Proposition C.2 lead naturally to:

**Corollary 1.** *The Markov chain* $\{X_t; t = 1, 2, \ldots\}$ *generated by the S.R.S in eq. (C.2) is positive Harris recurrent.*

*Proof.* This is Theorem C9.3(ii) in (Bhattacharya and Majumdar, 2007). □

**Remark D.2.** *Corollary 1 implies, in particular, that* $\pi(\mathcal{X}) < \infty$*. If we can now prove that the Markov chain* $\{X_t; t = 1, 2, \ldots\}$ *is also aperiodic, we can appeal to standard results for convergence and existence of moments. This can be established by an appeal to the results in p.293 of Stachurski (2009, 11.3.5). In particular, the function* $g$ *defined there is directly applicable to our case, as can be verified by the interested reader.*

**Remark D.3.** *We are now prepared to show that our key result follows from standard theorems on convergence of moments for Markov chains.*

*(i) The second of the two questions to be answered,* $\int_{\mathcal{X}} \log(x)\pi(dx) < \infty$*, follows from Remark D.2 and Condition 4. In other words, both the first moment of* $\{X_t; t = 1, 2, \dots\}$ *and the logarithmic moment are finite.*

*(ii) The first question, on the convergence of logarithmic moments, can also be answered in the affirmative. Given the finiteness of* $\pi$*, any non-degenerate initial distribution,* $\psi$*, for* $\{X_t; t = 1, 2, \dots\}$*, implying a non-degenerate distribution for time* $t$*,* $\psi_t$*, should satisfy the conditions in Tweedie (1983, Theorem 2). Thus, from Tweedie (1983, Theorem 2), it is the case that* $\lim_{t\to\infty} \mathbb{E}_{\psi_t}(\log(X_t)) = \mathbb{E}_{\psi}(\log(X)) := \int_S \log(x)\pi(dx)$