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Climate Change Policy under Spatial Heat Transport and Polar Amplification

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Climate Change Policy under Spatial Heat Transport and Polar Amplification

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Abstract

This paper is, to our knowledge, the first paper in climate economics to consider the combination of spatial heat transport and polar amplification. We simplified the problem by stratifying the Earth into latitude belts and assuming, as in North et al. (1981), that the two hemispheres were symmetric. Our results suggest that it is possible to build climate economic models that include the very real climatic phenomena of heat transport and polar amplification, and still maintain analytical tractability. We demonstrate the importance of heat transfer and polar amplification in the welfare analysis of climate change, and in particular on the social price of the climate change externality. Furthermore, we show that the effect of heat transfer and polar amplification on climate policy depend upon the interaction of climate component dynamics with the distribution of welfare weights, population, and productive capacities across latitudes. We discuss optimal fossil fuel taxes in a competitive environment with income effects and show that optimal taxes have a spatial structure and are dependent on each latitude's output. In addition, we characterize the interactions between spatial transport phenomena and the competitive equilibrium price path of tradable permits. Using general power utility functions, we show that an increase in the coefficient of relative risk aversion will reduce the social price of the climate externality.

JEL Classification: Q54, Q58, C61

Keywords: climate change, heat transport, polar amplification, welfare maximization, fossil fuels, optimal taxation, emissions permits

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1 Introduction

While spatial heat transport and polar amplification are well-established phenomena in the science of climate change, they have been largely ignored in the economic modeling of climate change. This paper introduces spatial heat transport and polar amplification in a simple spatial climate economics model, in which the climate model is based upon the work of Alexeev et al. (2005) and Langen and Alexeev (2007).

In this work the strength of spatial poleward heat transport from the lower latitudes to the higher latitudes depends upon the level of global mean average temperature. The spatial transport effect causes polar amplification due to increased meridional latent heat transport, as discussed by Alexeev et al. (2005) and further developed by Alexeev and Langen (2007) and Alexeev and Jackson (2012).

In order to exhibit the economic and climatic effects of spatial heat transport in the clearest and simplest possible way, we stratify the Earth into latitude belts and model the change in damages to each latitude belt from increased CO_2 into the atmosphere resulting from fossil fuel use in economic production activities located at each latitude. In order to focus completely on spatial climatic heat transport, we assume that total production at latitude belt x is given by $y(x,t)\,E(x,t)^{\alpha}$, $0<\alpha<1$, where $y(x,t)$ grows exogenously and $E(x,t)$ denotes emissions from fossil fuel inputs, or fossil fuel use by an appropriate choice of units in total production. This simplification and abstraction away from the allocative effects of other inputs to production on the economic side of the model enables us to keep a tight focus on climatic heat transport effects. In this context our approach and contributions can be explained in the following way.

First, consider the usual welfare optimization problem in which a social planner chooses the latitude emissions to maximize the integral over latitudes and time of discounted weighted utilities of consumption per capita where the climate dynamics are modeled by an energy balance model with spatial heat transport. Progress in this kind of modeling of more realistic climate representations in Integrated Assessment Models (IAMs) has been hindered by the analytical difficulties in dealing with more realistic climatic heat and moisture transport dynamics across a continuum of locations, and the modeling of the carbon cycle under anthropogenic forcing.

Seeking more realistic climate representations, we follow recent research suggesting that global temperature change can be described by a cumulative carbon emission budget. In this context, global mean warming is linearly proportional to cumulative carbon[1] with the slope of the linear relationship referred to as the transient climate response to cumulative CO_2 emissions (TCRCE).[2] We show that by using the cumulative carbon emission budget approach, and by expanding the climate dynamics of the latitude temperature field $T(x,t)$ into an infinite series of even numbered Legendre polynomials, the optimization model can be solved to any desired degree of accuracy for usual specifications of utility functions and latitude climatic damages. This analytical contribution enables economists to introduce climate effects of spatial transport and still retain some useful analytical tractability in climate economic models at this level of aggregation. We believe that this theoretical contribution is important for advancing analytically tractable IAM modeling because it introduces more realistic climate dynamics than, for example, simple three box carbon cycle models and two box temperature dynamics models, for the climate component of IAMs. Analytic tractability enables us to understand how the climate and economic components of an IAM interact to produce outcomes.

As an example, consider the case of zero income effects when it is optimal for the tax function to be uniform across latitudes. In this case, we show that an increase in the strength of poleward amplification of transport of heat energy $\hat{r}$ from $\hat{r} = 0$ to a small positive number causes the optimal tax function to shift upward or downward, depending on the interaction of the distribution of welfare weights, population, and damages per capita across latitudes with the distribution of the temperature anomaly $T(x,t)$ across latitudes at each point in time. We illustrate this marginal distribution effect of increased polar amplification with data and plots of population distribution data across latitudes for potentially plausible per capita damage

[1] As stated by Pierrehumbert (2014, p. 346), "CO_2 radiative forcing is concave downward as a function of concentration. However, the air fraction nonlinearity makes CO_2 concentration concave upward as a function of cumulative emissions, and it is a somewhat fortuitous consequence of the nature of carbonate chemistry nonlinearities that these two nonlinearities very nearly cancel for cumulative emissions up to several thousand gigatonnes of carbon. As a result, the value of a change in radiative forcing ΔF at the end of a given time period is linearly proportional to the cumulative emissions during that interval."

[2] See Matthews at al. (2009), Pierrehumbert (2012-2013, 2014), Matthews et al. (2012), MacDougall and Freidlingstein (2015), Leduc et al. (2016).

distributions across latitudes.[3]

It is worth noting that our analysis indicates that ignoring heat transport and polar amplification in the standard economic models of climate change implies a potential bias in the calculation of emission taxes and emission path for policy purposes. The present work on distributional impacts of climate change suggests that it is worthwhile to generalize IAMs to include marginal distributional impacts of spatial heat and moisture transport across latitudes and longitudes. We believe that isolating this marginal impact of polar amplification of heat transport on the optimal tax function is new in our paper.

Desmet and Rossi-Hansberg (2015) have studied spatial effects in IAM models at the level of disaggregation into latitude belts as we do in this paper. They do not include heat transport effects across latitudes and do not include polar amplification effects. On the other hand, they include the important adaptation response of migration to negative climate change while we do not include this response. Their paper shows the importance of removing, or at least reducing, restrictions on the adaptive response of migration.

Second, in a world where compensatory transfers are not possible, the usual result - that emission taxes should be uniform - fails because of income effects. We show that "poorer" latitudes should be taxed less than "richer" latitudes due to income effects. Furthermore we conduct comparative dynamics of optimal emissions taxes w.r.t. parameters, e.g. the strength of heat transfer, the strength of polar amplification $\hat{r}$, due to increased poleward latent heat transport, and more. Our comparative dynamics indicate that the optimal tax function depends not only on socioeconomic factors, but also on the interactions of these factors with climate dynamics as they are reflected in the heat transport process.

Third, our decomposition of the temperature field $T(x,t)$ into modes enables us to rank the modes by response times with the higher numbered modes responding faster than the lower numbered modes. This decomposi-

[3] E.g. we might expect that poorer latitudes will experience larger per capita damages, all other things equal (Burgess et al. (2014), Dell et al. (2012)). For example, Dell et al. (2012), have stressed the damaging effects of climatic changes in temperature and precipitation upon not only output levels but also growth rates of poorer countries. Burgess et al. (2014) document increased death rates due to high temperature extremes among the poor who do not have access to adaptation strategies such as air conditioning.

tion allows us to show that optimal paths may induce polar amplification.

The remainder of this paper is organized as follows. Section 2 develops the basic analytical framework used in the paper. Section 3 conducts welfare analysis and derives optimality conditions for the unified spatial climate and economic model. Section 4 studies the impact and exhibits the importance of heat transfer and polar amplification in the welfare analysis of climate change, and in particular on the social price of the climate change externality. This section shows how the comparative dynamics of heat transfer strength and polar amplification strength depend upon the interaction of climate component dynamics with the distribution of welfare weights, population, and productive capacities across latitudes. Section 5 discusses optimal fossil fuel taxes in a competitive environment with income effects; we show that optimal taxes have a spatial structure and are dependent on each latitude's output. We view our contribution here to be a type of "second best" analysis that moves the discussion of optimal taxation in climate economics towards adding more realistic institutional constraints on compensatory transfers across different sovereigns, as well as moving the discussion of optimal taxation towards more realistic climate models, but still preserving analytical tractability. Chichilnisky and Sheeran (2009) and Chichilnisky (2015) argue that efficiency and equity in a world of independent sovereigns could be tackled in climate economics by first bargaining over the allocation of emission permits, and then opening a trading market in these permits. We show how spatial transport phenomena interacting with the permit allocation impacts the competitive equilibrium price path of tradable permits. We conduct an analysis of permit prices as well as permit wealth ratios across latitudes and show how spatial transport impacts these quantities of economic interest. Section 6 conducts the same type of analysis as was done for logarithmic utility in earlier sections, but for general power utility functions. We show that an increase in the coefficient of relative risk aversion will reduce the social price of the climate externality. Section 7 includes a short summary, conclusions, and suggestions for future research. The Appendix contains the proofs of the propositions.

2 Temperature Dynamics and Heat Transport

To study the evolution of local temperature and its impact on climate policy when heat transport across the globe is taken into account, we build and extend the standard one-dimensional energy balance model (EBM) developed by North (1975a,b), North et al. (1981), and Wu and North (2007). We also substantially extend the work of Brock et al. (2013, 2014) which, for the first time to our knowledge, introduced into an one-dimensional EBM with spatial heat transport, the anthropogenic influence on local temperature resulting from the accumulation of carbon in the atmosphere and conducted economic optimization analysis in this type of model.

Let x denote the sine of the latitude. For simplicity we will just refer to x as "latitude", and let $T_{total}(x,t)$ denote surface (sea level) temperature measured in C° at latitude x and time t. We assume constant albedo across latitudes. The simplifying assumption of constant albedo allows us to cancel out the solar input and the constant in the outgoing radiation term of North et al. (1981) and decompose $T_{total}(x,t)$ into two parts: a baseline part and the temperature anomaly which is associated with human actions. Thus we define surface temperature as:

$$T_{total}(x,t) \equiv T_b(x,t) + T(x,t), \tag{1}$$

where the baseline temperature $T_b(x,t)$ is what the temperature at (x,t) would have been if humans were not increasing the carbon content of the atmosphere beyond the pre-industrial levels, and $T(x,t)$ is the temperature anomaly, which is the temperature increase attributed to the anthropogenic emissions of greenhouse gasses (GHGs).

The basic energy balance equation for the surface temperature with human input added can be written as:

$$C\frac{\partial T_{total}(x,t)}{\partial t} = Q(x,t) - (A + BT_{total}(x,t) +) D\mathcal{L}T_{total}(x,t) + \Delta F \tag{2}$$

$$T_{total}(x,0) = T_b(x,0), \text{ given}$$

$$\mathcal{L}T_{total}(x,t) \equiv \frac{\partial}{\partial x}\left[\frac{\left(1-x^2\right)\partial T_{total}(x,t)}{\partial x}\right], \tag{3}$$

where $x = 0$ denotes the Equator, $x = 1$ denotes the North Pole and $x = -1$ denotes the South Pole, and the heat capacity parameter "C" of North et al.

(1981) is absorbed into the other parameters of (2). That is, we put $C = 1$ by absorbing it into the other parameters in (2). In (2), $Q(x,t)$ is the solar forcing, and D is a heat transport coefficient which is an adjustable parameter measured in W/(m^2)(°C) which has been calibrated to match observed temperatures across latitudes. This coefficient can also be expressed in dimensionless form as in North et al. (1981). Finally ΔF denotes radiative forcing associated with anthropogenic emissions $E(t)$ of GHGs. Alexeev et al. (2005) specify the heat transport coefficient as a function of the temperature anomaly as[4]

$$D(T_M(t)) = D_{ref}(T_M(t) - T_{ref}), \tag{4}$$

where $T_{ref} = 15°C$ and T_M is global mean temperature. Setting $T_M(t) = T_b + \bar{T}(t)$ and $T_b = T_{ref}$ we obtain

$$D(T_M(t)) = D_{ref}[1 + \hat{r}(T_b(t) + T(t) - T_{ref}(t))] = D_{ref}\left[1 + \hat{r}\bar{T}(t)\right] \tag{5}$$

$$D_{ref} = 0.445, \;\; \hat{r} = 0.03/K. \tag{6}$$

The operator $\mathcal{L}$ is a linear operator on the space of functions of x with the property that the nth Legendre polynomial $P_n(x)$ is an eigenfunction of $\mathcal{L}$, i.e. $\mathcal{L}P_n(x) = -\lambda_n P_n(x)$, $\lambda_n = n(n+1)$.[5] We use this property in the solution of the model. The term $D(T(t))\mathcal{L}T(x,t)$ therefore models the heat flux associated with the temperature anomaly.

To enhance the tractability of the optimized model, since in (2) dynamics are described by partial differential equations (PDEs) with a nonlinear diffusion term, we introduce two approximations, one from North et al. (1981) and the other from Matthews et al. (2009).

North et al. (1981) note that $T(x,t)$ can be written in a series expansion in terms of Legendre polynomials, or

$$T(x,t) = \sum_{n=0,\text{ even}} T_n(t) P_n(x) \tag{7}$$

where $P_n(x)$ is the nth Legendre polynomial. They approximate $T(x,t)$ by

[4] Alexeev et al. (2005) specify $\hat{r}$ to be 3% per degree Kelvin.

[5] $P_n(x) = 2^n \sum_{k=0}^{n} \binom{n}{k} \binom{n+k-1/2}{n}$, $P_0(x) = 1$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$.

truncating the expansion at some finite N.[6] This implies that the average global temperature anomaly (7) can be defined as:

$$\bar{T}(t) \equiv \frac{1}{2}\int_{-1}^{1}\sum_{n=0}^{\infty} T_n(t) P_n(x)\, dx = T_0(t). \tag{8}$$

Then, following Alexeev et al. (2005) the heat transport coefficient can be defined as:

$$D(T_M(t)) = D\left[1 + \hat{r}T_0(t)\right]. \tag{9}$$

Following Matthews et al. (2009) and MacDougall and Freidlingstein (2015), Leduc et al. (2016)[7] the radiative forcing ΔF is linearly proportional to emissions at date t, in order that the addition to global mean temperature at date t is approximately proportional to cumulative emissions at date t, or:

$$\Delta F = \lambda E(t) \ \ , \ E(t) = \int_{x=-1}^{x=1} E(x,t)\, dx \tag{10}$$

Using North's approximation for the baseline local temperature and the local temperature anomaly which is

$$T_b(x,t) = \sum_{n,\text{ even}} T_{bn}(t) P_n(x) \ \ , T(x,t) = \sum_{n,\text{ even}} T_n(t) P_n(x), \tag{11}$$

respectively we can write the total temperature dynamics (baseline plus anomaly) as:

$$\begin{aligned}\frac{\partial (T_b(x,t) + T(x,t))}{\partial t} &= \sum_{n=0}^{\infty}\left(\dot{T}_n(t) + \dot{T}_{bn}(t)\right) P_n(t) = \\ Q(x,t) &- \left(A + B\sum_{n=0}^{\infty}(T_n(t) + T_{bn}(t)) P_n(x)\right) + \\ \left[D\left(1 + \hat{r}\left(\bar{T}_b(t) + \bar{T}(t) - 15\right)\right)\right] &\mathcal{L}(T + T_b)(x,t) + \Delta F\end{aligned} \tag{12}$$

[6] Note that $\sum_{n=0}^{\infty} T_n(0) P_n(0)$ does not imply that all $T_n(0)$'s are zero. Indeed, if all $T_n(0)$'s are zero, then the solution of (2) would be independent of x and all spatial effects would vanish for the anomaly. As one might expect, if one is dealing with differential equations in an infinite dimensional space, an infinite number of initial conditions must be specified.

[7] Leduc et al. (2016) show that regional temperatures also respond approximately linearly to cumulative CO2 emissions. We thank Victor Zhorin of RDCEP, University of Chicago, for bringing this paper to our attention.

The baseline temperature dynamics are given by

$$\frac{\partial T_b(x,t)}{\partial t} = \sum_{n=0}^{\infty} \dot{T}_b(t) P_n(t) = \tag{13}$$

$$Q(x,t) - \left(A + B\sum_{n=0}^{\infty} T_b(t) P_n(x)\right) + \left[D\left(1+\hat{r}\left(\bar{T}_b(t) - 15\right)\right)\right] \mathcal{L}(T_b)(x,t)$$

$$= \quad Q(x,t) - \left(A + B\sum_{n=0}^{\infty} T_{bn}(t) P_n(x)\right) + D\mathcal{L}(T_b)(x,t),$$

using $\bar{T}_b(t) = T_{ref} = 15$. Taking the difference between total and baseline temperature dynamics, we obtain the temperature anomaly dynamics as:

$$\frac{\partial T(x,t)}{\partial t} = \sum_{n=0}^{\infty} \dot{T}_n(t) P_n(t) = -B\sum_{n=0}^{\infty} T_n(t) P_n(x) \tag{14}$$

$$+D\left(1+\hat{r}\bar{T}(t)\right) \mathcal{L}(T+T_b)(x,t) - D\mathcal{L}(T_b)(x,t) + \Delta F$$

$$= \quad -B\sum_{n=0}^{\infty} T_n(t) P_n(x) + D(1+\hat{r}T_0(t))\mathcal{L}T(x,t) + D(1+\hat{r}T_0(t))\mathcal{L}T_b(x,t)$$

$$-D\mathcal{L}T_b(x,t) + \Delta F.$$

Using $\mathcal{L}P_n(x) = -\lambda_n P_n(x)$, $\lambda_n = n(n+1)$ for the eigenvalues of $\mathcal{L}(\cdot)$, we obtain:

$$\frac{\partial T(x,t)}{\partial t} = \sum_{n=0}^{\infty} \dot{T}_n(t) P_n(x) = -B\sum_{n=0}^{\infty} T_n(t) P_n(x) \tag{15}$$

$$-D\left[\sum_{n=0}^{\infty} \lambda_n T_n(t) P_n(x)\right] + \Delta F - \hat{r}DT_0(t)\left(\sum_{m=0}^{\infty} \lambda_m \left[T_{bm}(t) + T_m(t)\right] P_m(t)\right)$$

$$T_0(x,0) = \sum_{n=0}^{\infty} T_n(0) P_n(x) = 0 \text{ , given.}$$

We can simplify (15) by using the property that the Legendre polynomials are orthogonal with respect to the inner L^2 product on the interval

$x \in [-1, 1]$, which implies that

$$\langle P_n, P_m \rangle = \int_{x=-1}^{x=1} P_n(x) P_m(x) dx = \frac{2}{2n+1} \delta_{nm} \tag{16}$$

$$\delta_{nm} = \begin{cases} 1 & \text{if} \quad n = m \\ 0 & \text{if} \quad n \neq m. \end{cases} . \tag{17}$$

Therefore, multiplying both sides of (15) by $P_n(x)$, integrating over $x \in [-1, 1]$ (16-17), using the inner product notation $\int_{x=-1}^{x=1} F(x) G(x) dx = \langle F, G \rangle$, noting that $P_0(x) = 1$, $\int_{-1}^{1} P_0(x) dx = 2$, $\int_{-1}^{1} P_n(x) dx = 0$, $n = 2, 4, 6, \ldots$, and dropping the higher order term

$$\hat{r} D T_0(t) \sum_{m=0}^{\infty} \lambda_m T_m(t) P_m(t) = o(\|T(\cdot)\|) \tag{18}$$

in the norm of the function $T(\cdot)$, we obtain:[8]

$$\dot{T}_n(t) = [-B - D\lambda_n] T_n(t) + \Delta F \frac{\langle 1, P_n \rangle}{\langle P_n, P_n \rangle} - \hat{r} D T_0(t) \lambda_n T_{bn}(t) \tag{19}$$
$$n = 0, 2, 4, \ldots \quad .$$

A two-mode approximation of (19), for example, results in the following system of ordinary differential equations:

$$\dot{T}_0(t) = -BT_0(t) + \lambda E(t) \ , \ P_0(x) = 1, \langle P_0, P_0 \rangle = 2 \tag{20}$$

$$\dot{T}_2(t) = (-B - 6D) T_2(t) - 6\hat{r} D T_0(t) T_{b2}(t) \tag{21}$$

$$\langle 1, P_2 \rangle = 0, \ P_2(x) = \frac{1}{2}\left(3x^2 - 1\right). \tag{22}$$

We will use temperature dynamics (19) to derive the optimal emission paths and the corresponding optimal spatial taxes.

[8] Since the anomalies in the modes, $T_0(t), T_2(t), \ldots$ are small, we expect the products to be small enough that the optimal paths ignoring the second order term (18) are workably close to the optimal paths when (18) is not dropped. However, numerical work is needed to actually verify how large the error in the optimal path is when (18) is dropped.

3 Welfare Maximization under Heat Transfer

To study optimal emissions paths in the context of the one-dimensional climate model described above, we consider a simple welfare maximization problem with logarithmic utility, where world welfare is given by:[9]

$$\int_{t=0}^{\infty} \mathrm{e}^{-\rho t} \left[\int_{x=0}^{x=1} v\left(x\right) L\left(x\right) \ln \left[y\left(x,t\right) E\left(x,t\right)^{\alpha} \left(1 - A\left(x,t\right)\right) \mathrm{e}^{-\phi_T(x)(T_{total} - bA)} \right] \right] dxdt, \tag{23}$$

where $y\left(x,t\right) E\left(x,t\right)^{\alpha}$, $0 < \alpha < 1$, $E\left(x,t\right)$, $T\left(x,t\right)$, $L\left(x\right)$ are output per capita, fossil fuel input, temperature anomaly, and fully employed population at location (or latitude) x at date t, respectively. The term $yE^{\alpha}\left(1 - A(x,t)\right)$ stands for the fraction of output available for consumption after adaptation. The term $\mathrm{e}^{-\phi_T(x)[T_{total}(x,t) - bA(x,t)]}$ reflects damages to output per capita in location x from an increase in the temperature anomaly at this location, which is the term $\mathrm{e}^{-\phi_T(x) T_{total}(x,t)}$ net of reduction in damages due to adaptation, which is the term $\mathrm{e}^{\phi_T(x) bA(x,t)}$. The damage coefficient ϕ_T may depend upon time, i.e. $\phi_T = \phi_T\left(x,t\right)$. Note that damages depend upon total temperature at (x,t) which is defined by the sum of baseline temperature which would have occurred if there were no human emissions into the system, $T_b\left(x,t\right)$, and the temperature anomaly, $T\left(x,t\right)$, caused by human emissions into the atmosphere. We assume that $y\left(x,t\right)$, $L\left(x\right)$ are exogenously given and fixed. That is, we are abstracting away from the problem of optimally accumulating capital inputs and other inputs in order to focus sharply on optimal fossil fuel taxes under transport effects. Finally, $v\left(x\right)$ represents welfare weights associated with location x.

Formulation (23) allows the incorporation of another very important aspect of spatially distributed damages from climate change, namely damages from precipitation. Defining total precipitation as the sum of baseline precipitation and the precipitation anomaly or $P_{total}\left(x,t\right) = P_b\left(x,t\right) + P\left(x,t\right)$, Castruccio et al. (2014) suggest the following approximation for the precipitation anomaly:[10]

$$P\left(x,t\right) = \psi\left(x\right) T\left(x,t\right). \tag{24}$$

Assuming exponential precipitation damages of the form $\exp\left(-\varphi\left(x,t\right)\left(P_b\left(x,t\right) + P\left(x,t\right)\right)\right)$

[9] For the rest of the paper we follow North (1975a,b) and North et al. (1981) and consider the northern hemisphere only, i.e. $x \in [0,1]$.

[10] We ignore the conditional variance since we are working with a deterministic model.

and using Castruccio et al.'s (2014) approximation, we can write a welfare function that contains both temperature impacts and precipitation damages as:

$$\int_0^\infty \mathrm{e}^{-\rho t} \int_0^1 \left[v(x) L(x) \ln\left(y(x,t) E(x,t)^\alpha\right) \times \right. \tag{25}$$
$$\left. \left(\mathrm{e}^{-\phi_T(x,t)[T_b(x,t)+T(x,t)]} \mathrm{e}^{-\varphi(x,t)[P_b(x,t)+\psi(x)T(x,t)]} \right) \right] dxdt.$$

In the case where we are assuming logarithmic utility and exponential damages to output both from temperature and precipitation, we can add a baseline temperature $T_b(x,t)$ and a baseline precipitation $P_b(x,t)$ to the corresponding anomalies and still be able to assert that (25) can be replaced for optimization purposes by the equivalent problem,

$$\max_{E(x,t)} \left\{ \int_0^\infty \mathrm{e}^{-\rho t} \int_0^1 v(x) L(x) \left\{ \alpha \ln \left[E(x,t)(1-A) \right] - \phi(x) \left[T(x,t) - bA \right] \right\} dxdt \right\}, \tag{26}$$

where $\phi(x,t) = \phi_T(x,t) + \varphi(x,t)\psi(x)$. In the definition of $\phi(x,t)$, the term $\phi_T(x,t)$ accounts for temperature damages, while the term $\varphi(x,t)\psi(x)$ allows for precipitation damages. It should be noted that to the best of our knowledge, this is the first time that climate economics in terms of a spatial one-dimensional EBM that incorporates the important climate science phenomenon of heat transfer is combined with the spatial characteristics of damages from temperature and precipitation. This combination results in a model of climate economics capable of determining the impact on the social cost of climate externality of including spatial heat transport and, hence, the impact of spatial heat transport on optimal fossil fuel taxes.

The problem of a social planner would be to choose fossil fuel paths $E(x,t)$ or equivalently, by an appropriate change in units, emissions paths $E(x,t)$ to maximize (26) subject to climate dynamics given by (19), and an additional constraint reflecting the potential exhaustibility of global fossil fuel reserves.

$$\int_{t=0}^\infty E(t)\, dt < R_0 \ , \ E(t) = \int_{x=0}^{x=1} E(x,t)\, dx \ , \ \int_{x=0}^{x=1} R_0(x) = R_0, \tag{27}$$

where R_0 denotes global fossil fuel reserves, and $R_0(x)$ fossil fuel reserves in location x.

Constraint (27) implies that the social planner is altruistic and treats fossil fuels reserves as a common property which can be transferred across locations. The alternative polar case is to assume that no transfers are possible and that each location is constrained by local fossil fuel reserves, or

$$\int_{t=0}^{\infty} E(t,x)\,dt < R_0(x) \quad \text{for all } x \in [0,1] \ , \quad \int_{x=0}^{x=1} E(x,t)\,dx = E(t)\,. \tag{28}$$

We start with the welfare maximization problem of the altruistic planner, making the simplifying assumption that the damage parameter $\phi(x,t)$ is independent of t. The current value Hamiltonian for this problem is:

$$\begin{aligned} & H = \\ & \int_{x=0}^{x=1} \left\{ v(x)\,L(x) \left[\ln\left[E^{\alpha}(1-A)\right] - \phi(x) \left[\sum_{n=0,2,\ldots} T_n(t)\,P_n(x) - bA \right] \right] \right. \\ & \left. -\lambda_R(t)\,E(t,x)\,dx \right\} + \\ & \sum_{n=0,2,\ldots} \lambda_{T_n} \left[\left[-B - D\lambda_n\right] T_n(t) + \lambda \left(\int_0^1 E(x,t)\,dx \right) \delta_{n0} - \hat{r} D T_0(t)\,\lambda_n T_{bn}(t) \right], \end{aligned} \tag{29}$$

where $\delta_{n0} \equiv \frac{\langle 1,P_n \rangle}{\langle P_n,P_n \rangle}$, Note that $\delta_{n0} = 0, n \neq 0$, $\delta_{n0} = 1, n = 0$. The two-mode approximation, for example, would result in the following current value Hamiltonian:

$$\begin{aligned} & H = \\ & \int_{x=0}^{x=1} \left\{ v(x)\,L(x) \left[\ln\left[E^{\alpha}(1-A)\right] - \phi(x) \left[T_0(t) + T_2(t)\,P_2(x) - bA \right] \right] \right. \\ & \left. -\lambda_R(t)\,E(t,x) \right\} dx + \\ & \lambda_{T_0}(t) \left[-BT_0(t) + \lambda E(t) \right] + \\ & \lambda_{T_2}(t) \left[(-B - 6D)\,T_2(t) - 6\hat{r} D T_0(t)\,T_{b2}(t) \right]. \end{aligned} \tag{30}$$

The first order necessary conditions (FONC) resulting from the maximum principle, after suppressing the (x,t) arguments to ease notation when necessary, can be obtained as follows. The optimal emission (or fossil fuel)

$E^*(x,t)$ path and optimal adaptation $A^*(x)$ satisfy:

$$\frac{\alpha v(x) L(x)}{E^*(x,t)} = \lambda_R(t) - \lambda \lambda_{T_0}(t) \implies \tag{31}$$

$$E^*(x,t) = \frac{\alpha v(x) L(x)}{\lambda_R(t) - \lambda \lambda_{T_0}(t)}, \tag{32}$$

$$\frac{1}{1 - A^*(x)} = b\phi(x) \implies A^*(x) = \frac{b\phi(x) - 1}{b\phi(x)}. \tag{33}$$

In (31), $\xi_C(t) = -\lambda \lambda_{T_0}(t)$ is the social price of the climate externality and $\xi_F(t) = \lambda_R(t) - \lambda \lambda_{T_0}(t)$ is the social price of fossil fuels. Here we define social price of the climate externality to allow it to be negative, which it usually will be since it is typically a "bad". Furthermore (33) implies that for all latitudes and all dates where $b\phi(x) - 1 > 0$, there will be adaptation expenditures.

Setting $\mathrm{d}(x) = v(x) L(x) \phi(x)$ to simplify the exposition, the costate variables evolve according to

$$\dot{\lambda}_{T_0} = (\rho + B) \lambda_{T_0} + \sum_{n=0,2,...} \lambda_{T_n} n(n+1) \hat{r} D T_{bn}(t) + \langle 1, \mathrm{d}(x) \rangle \tag{34}$$

$$\dot{\lambda}_{T_n} = (\rho + B + Dn(n+1)) \lambda_{T_n} + \langle P_n(x), \mathrm{d}(x) \rangle \ , n = 2, 4, ... \tag{35}$$

$$\dot{\lambda}_R(t) = \rho \lambda_R(t), \tag{36}$$

while temperature dynamics are given by

$$\dot{T}_0 = -BT_0(t) + \lambda E^*(t) \tag{37}$$

$$\dot{T}_n = -[B + D\lambda_n] T_n(t) - \hat{r} D T_0(t) \lambda_n T_{bn}(t) \ \ n = 2, 4, ... \tag{38}$$

and the fossil fuel constraint satisfies

$$E^*(t) = \int_{x=0}^{x=1} E^*(x,t) \, dx \ , \ \int_{t=0}^{\infty} \langle 1, E^*(x,t) \rangle dt = R_0. \tag{39}$$

If we assume that each location is constrained by local fossil fuel reserves $R_0(x)$ and that no transfers are possible, condition (36) should be replaced by

$$\dot{\lambda}_R(x,t) = \rho \lambda_R(x,t). \tag{40}$$

Then

$$E^{*}(x,t)=\frac{\alpha v(x)L(x)}{\lambda_{R}(x,t)-\lambda\lambda_{T_0}(t)},\tag{41}$$

while the fossil fuel constraint becomes

$$\int_{x=0}^{x=1}E^{*}(x,t)\,dx=R_{0}(x).\tag{42}$$

For the two-mode approximation the costate variables evolve according to

$$\begin{aligned}\dot{\lambda}_{T_0} &= (\rho+B)\lambda_{T_0}+6\hat{r}DT_{b2}(t)\lambda_{T_2}+\langle 1,\mathrm{d}\rangle &(43)\\ \dot{\lambda}_{T_2} &= (\rho+B+6D)\lambda_{T_2}+\langle P_2,\mathrm{d}\rangle. &(44)\end{aligned}$$

3.1 Welfare Maximization when Heat Transfer is Ignored

To understand the impact of heat transfer on optimal fossil fuel paths (or emission paths) and optimal climate policy, it is helpful to consider at the beginning welfare optimization where heat transfer is ignored, or $D=0$. The optimality conditions (34-36) with $D=0$ become:[11]

$$\begin{aligned}\dot{\lambda}_{T_0} &= (\rho+B)\lambda_{T_0}+\langle 1,\mathrm{d}\rangle &(45)\\ \dot{\lambda}_{T_n} &= (\rho+B)\lambda_{T_n}+\langle P_n,\mathrm{d}\rangle\ ,n=2,4,... &(46)\end{aligned}$$

while temperature dynamics in (37-38) are independent of D. Taking the forward solutions for the costate variables we obtain:

$$\begin{aligned}\lambda_{T_0} &= -\int_{s=0}^{\infty}e^{-(\rho+B)(s-t)}\langle 1,\mathrm{d}\rangle ds &(47)\\ \lambda_{T_n} &= -\int_{s=0}^{\infty}e^{-(\rho+B)(s-t)}\langle P_n,\mathrm{d}\rangle ds. &(48)\end{aligned}$$

Then the optimal fossil fuel path for the log utility case is given by

$$E^{*}(x,t)=\frac{\alpha v(x)L(x)}{\lambda_{R}(0)e^{\rho t}-\lambda\lambda_{T_0}(t)}.\tag{49}$$

[11]Note that $D>0$ implies that spatial heat transport occurs. On the other hand, we need $\hat{r}>0$ so that spatial heat transport generates impacts in the context of our model, that is, to create asymmetric effects such as polar amplification. Note that D appears in mode 2's co-state, but $\hat{r}=0$ removes the effect of mode 2's costate on the dynamics of mode zero's costate, as can be seen from (43)-(44).

Using the assumption that population and the damage parameter do not change with time, the steady-state values for the costate variables implied from (45-46) are:

$$\lambda^*_{T_0} = -\frac{\langle 1, \mathrm{d} \rangle}{(\rho + B)}, \lambda^*_{T_n} = -\frac{\langle P_n, \mathrm{d} \rangle}{(\rho + B)} \tag{50}$$

$$\langle 1, \mathrm{d} \rangle = \int_{x=0}^{x=1} v(x) L(x) \phi(x) \, dx \tag{51}$$

$$\langle P_n, \mathrm{d} \rangle = \int_{x=0}^{x=1} P_n(x) v(x) L(x) \phi(x) \, dx. \tag{52}$$

This means that the steady-state costate variables are independent of location x. The resource constraint implies

$$R_0 \geq \int_{t=0}^{\infty} dt \int_{x=0}^{x=1} E(x,t) \, dx = \int_{t=0}^{\infty} \int_{x=0}^{x=1} \left(\frac{\alpha v(x) L(x)}{\lambda_R(0) e^{\rho t} - \lambda \lambda_{T_0}(t)} \right) dx dt. \tag{53}$$

The initial value $\lambda_R(0)$ can be obtained by solving (53) for this initial value for any given value of total reserves R_0. Conditions (50)-(52) and (53) completely determine the optimal emission path for each location with the population kept constant at each location. Since the steady-state costate variables are independent of location x, the social price of the climate externality and the social price of fossil fuels are independent of location x. If we consider the case in which each location is constrained by local fossil fuel reserves $R_0(x)$, and assume that no transfers are possible, then the local resource constraint implies

$$\int_{x=0}^{x=1} E^*(x,t) \, dx = \int_{x=0}^{x=1} \left(\frac{\alpha v(x) L(x)}{\lambda_R(x,0) e^{\rho t} - \lambda \lambda_{T_0}(t)} \right) dx = R_0(x). \tag{54}$$

This constraint can be used to determine the initial value $\lambda_R(x,0)$ for any given value of total local reserves $R_0(x)$. In this case, although the social price of the climate externality does not depend on the location, the social price of fossil fuels depends on location through local reserves. This result is similar to an analogous result in Brock et al. (2014).

4 Heat Transport and Climate Change Policy

We move now to one of the main objectives of this paper, which is the characterization of the impact of heat transport towards the Poles on the social price of climate externality $\xi_C(t) = -\lambda\lambda_{T_0}(t)$ and consequently on optimal fossil fuel paths and fossil fuel taxes. Since λ is a fixed parameter, the impact of heat transport should be realized through the costate variable λ_{T_0}. This costate determines the optimal tax for the correction of the climate externality. We determine the impact of spatial heat transport by the ratio

$$\psi \equiv \frac{\tau(\hat{r}=0)}{\tau(\hat{r}>0)} = \frac{\lambda_{T_0}(\hat{r}=0)}{\lambda_{T_0}(\hat{r}>0)}, \tag{55}$$

which is computed at the steady state of λ_{T_0}. Using $\mathrm{d}(x) = v(x) L(x) \phi(x)$, we obtain from (45)

$$\lambda_{T_0}(\hat{r}=0) = -\frac{\langle 1, \mathrm{d}(x)\rangle}{(\rho+B)}, \tag{56}$$

while from (34) we obtain

$$\lambda_{T_0}(\hat{r}>0) = -\frac{\langle 1, \mathrm{d}(x)\rangle + \sum_{n=2,\ldots} \lambda_{T_n} n(n+1)\hat{r} D T_{bn}(t)}{(\rho+B)} \tag{57}$$

$$\lambda_{T_n} = -\frac{\langle P_n(x), \mathrm{d}(x)\rangle}{(\rho+B+Dn(n+1))}, n = 2, 4, \ldots \ .$$

If we take the two-mode approximation $T_b(x,t) \cong T_{b0} + T_{b2} P_2(x)$ to the baseline temperature (North et al. 1981, equation (31)) with $T_{b0} = 14.97°\mathrm{C}$ and $T_{b2} = -28.0°\mathrm{C}$, then using (43) and (44),

$$\lambda_{T_0}(\hat{r}>0) = -\frac{\langle 1, \mathrm{d}(x)\rangle + 6\lambda_{T_2}\hat{r} D T_{b2}}{(\rho+B)}, \tag{58}$$

and the ratio (55) becomes

$$\frac{\lambda_{T_0}(\hat{r}=0)}{\lambda_{T_0}(\hat{r}>0)} = 1 \Big/ \left(1 + \frac{6\lambda_{T_2}\hat{r} D T_{b2}}{\langle 1, \mathrm{d}(x)\rangle}\right) \tag{59}$$

$$\lambda_{T_2} = -\frac{\langle P_2(x), \mathrm{d}(x)\rangle}{(\rho+B+6D)} = -\frac{\int_0^1 P_2(x) v(x) L(x) \phi(x)\, dx}{(\rho+B+6D)}. \tag{60}$$

The above analysis suggests the following proposition.

Proposition 1 *Ignoring spatial heat transport, i.e. setting* $\hat{r} = 0$ *when* $\hat{r} >$

0, *will lead to underestimation of the optimal climate externality tax if* $J > 0$, $J \equiv \frac{6\lambda_{T_2}\hat{r}DT_{b2}}{\langle 1, \mathrm{d}(x)\rangle}$. *The optimal climate externality tax will be overestimated if* $-1 < J < 0$.

The proof follows from the calculations above.

The value of the quantity of interest J depends on the distributions across latitudes of the welfare weights $v(x)$, the population $L(x)$, the damages $\phi(x)$ from an increase in global temperature, and $P_2(x)$ that reflects the dynamics of Nature on the spatial distribution of temperature. J can be written as

$$J \equiv \frac{6\hat{r}D(-T_{b2})\int_0^1 P_2(x)\, v(x)\, L(x)\, \phi(x)\, dx}{(\rho+B+6D)\int_0^1 v(x)\, L(x)\, \phi(x)\, dx} \tag{61}$$

$$\phi(x) = \phi_T(x) + \psi(x)\,\varphi(x), \tag{62}$$

or

$$J = \frac{6\hat{r}D(-T_{b2})}{(\rho+B+6D)}\int_0^1 P_2(x)\, s(x)\, dx \tag{63}$$

$$P_2(x) = \frac{1}{2}\left(3x^2-1\right),\ s(z) = \frac{v(z)\, L(z)\, \phi(z)}{\int_0^1 v(x)\, L(x)\, \phi(x)\, dx}, \tag{64}$$

where $s(x)$ can be interpreted as the share of weighted (by welfare weights $v(x)$) damages at location x. Since $T_{b2} < 0$, the sign of J is the sign of $\psi = \int_0^1 P_2(x)\, s(x)\, dx$. Let $\hat{r} = 0.03$, $D = 0.0445$, $T_{b2} = -28$, $\rho = 0.02$, $B = 2$ so that $\frac{6\hat{r}D(-T_{b2})}{(\rho+B+6D)} = 0.478$. If $s(x) = x$, $x \in [0,1]$, then $\psi = 0.05978$, and $J = 0.94359$. Thus when the share of damages is higher in higher latitudes, ignoring heat transport underestimates the optimal climate externality tax. On the other hand, if $s(x) = 1-x$, $x \in [0,1]$, then $\psi = -0.05977$ and $J = 1.06358$. Thus, in this example, when the share of damages is higher in lower latitudes, ignoring heat transport overestimates the optimal climate externality tax. In the case where $s(x)$ is a constant independent of x, then $J = 0$ since $\int_0^1 P_2(x)\, dx = 0$ and $\psi = 1$.

To obtain more insights regarding the potential values of J, we consider the general function

$$s(x) = \frac{\gamma(1-x)^{\alpha_0}(1+x)^{\beta_0}\left(\gamma_0+\delta_0 x^2\right)}{\int_{x=0}^{x=1}(1-x)^{\alpha_0}(1+x)^{\beta_0}\left(\gamma_0+\delta_0 x^2\right) dx}, \tag{65}$$

as an approximate distribution of $s(x)$ in $x \in [0,1]$. From (64) the share $s(x)$ depends on the distribution of welfare weights $v(x)$, population $L(x)$, and marginal damages $\phi(x)$. Work by Mendelsohn et al. (2006) or Burgess et al. (2014), for example, suggest that climate change is expected to be most severe in poor countries surrounding the equator, with a skew towards southern latitudes. If we follow the usual approach of setting welfare weights equal to Negishi-type weights, then these weights can be set according to GDP per capita across latitudes. See Kummu and Varis (2011) for data by latitudes. However equal weights across locations, or weights where the most importance is given to latitudes around the equator, are also possibilities. Finally, for the population, evidence suggests that roughly 88% of the world's population lives in the northern hemisphere, and about half the world's population lives north of 27°N.[12] The actual distribution of $s(x)$ is an empirical issue that requires further research. In order, however, to focus on the possible over- or under-estimation of the externality tax, we consider the alternative distributions shown in figure 1.

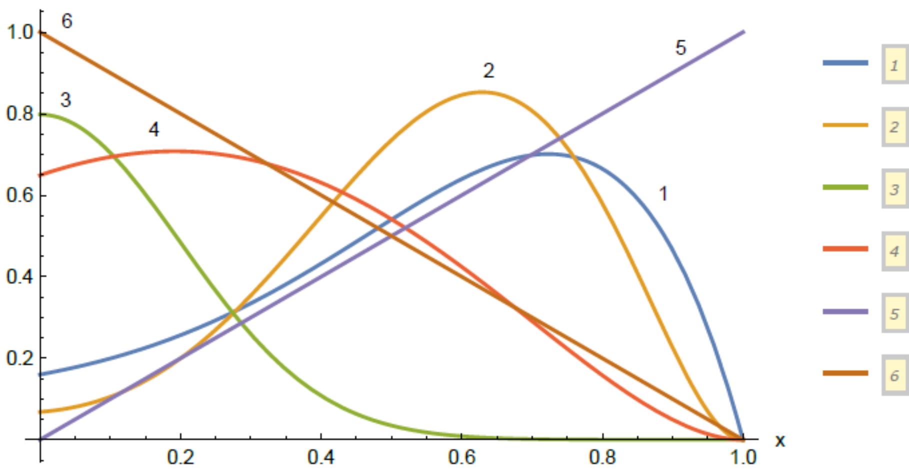

Figure 1: Possible $s(x)$ distributions.

In figure 1, distributions 1, 2 and 5 assign more damages to locations in the North while distributions 3, 4 and 6 assign more damages to locations

[12] See http://visual.ly/worlds-population-2000-latitude-and-longitude.

around the equator.

Table 1: Externality tax comparison

Distribution $s(x)$	J	$\psi = \frac{\lambda_{T_0}(\hat{r}=0)}{\lambda_{T_0}(\hat{r}>0)}$
1	0.01567	0.98456
2	0.00636	0.99368
3	-0.04208	1.04393
4	-0.052657	1.05558
5	0.05978	0.94359
6	-0.05978	1.06358

The results of table 1 show that ignoring heat transport will cause an underestimation of the optimal externality tax when the distribution of the weighted share of climate change damages is skewed towards the southern latitudes. The bias and its direction depends on natural parameters reflected in $P_2(x)$, but also on socioeconomic parameters reflected in the distribution of population, climate change damages and welfare weights. The important message, however, is that taking into account the spatial dynamics of nature emerging because of heat transport across latitudes, which is a well-documented natural process, changes the social price of the climate externality and the corresponding optimal tax relative to the case where heat transport is ignored. Since in general $\lambda_{T_0}(\hat{r}=0) \neq \lambda_{T_0}(\hat{r}>0)$, optimality conditions (32) or (41) suggest that optimal paths for fossil fuel use when heat transport across latitudes is ignored will in general either overestimate or underestimate the true optimal fossil fuel paths. That is, $E^*(x,t;\hat{r}>0) \neq E^*(x,t;\hat{r}=0)$.

4.1 Spatial Heat Transport and Cross Latitude Effects

Having established that taking into account that heat transport across latitudes affects the social price of the climate externality and consequently optimal emission paths and taxes through socioeconomic and natural factors, our next step is to examine in more detail the impacts of heat transport across locations on socially optimal fossil fuel use, the social price of fossil fuels, the socially optimal temperature paths and discount rate for future costs and benefits.

4.1.1 Fossil fuel use

In (32) or (41), assume that reserves are infinite so that $\lambda_R = 0$, and the optimal fossil fuel use path is given by[13]

$$E(x,t) = \frac{\alpha v(x) L(x)}{\lambda_R(x,0) \mathrm{e}^{\rho t} - \lambda \lambda_{T_0}(t)}. \tag{66}$$

The derivatives of $E(x,t)$ and $\lambda_{T_0}(t)$ with respect to $\hat{r}$ are denoted by $E'(x,t)$ and $\lambda'_{T_0}(t)$, and these derivatives are computed at $\hat{r} = 0$ so that the impact of taking into account heat transport ($\hat{r} > 0$) relative to ignoring it ($\hat{r} = 0$) is captured.

From (66)

$$E'(x,t) = \frac{-\alpha v(x) L(x) \left(\lambda'_R(x,0) \mathrm{e}^{\rho t} + \lambda \lambda'_{T_0}(t)\right)}{\left(\lambda_R(x,0) \mathrm{e}^{\rho t} - \lambda \lambda_{T_0}(t)\right)^2}. \tag{67}$$

Thus the sign of $E'(x,t)$ depends on the signs of $\lambda'_{T_0}(t)$ and $\lambda'_R(x,0)$. The derivative $\lambda'_{T_0}(t)$ was approximated in the previous section. An exact computation is given in the following proposition.

Proposition 2 *Under the two mode approximation of temperature dynamics, the impact on the shadow value of the mean global temperature from an increase in the heat transport across locations is given by*

$$\lambda'_{T_0}(t) = \frac{(6DT_{b2}) \langle P_2, \mathrm{d} \rangle}{2(\rho + B) + 6D} \mathrm{e}^{-[2(\rho+B)+6D]t}. \tag{68}$$

For the proof see Appendix.

Since following North et al. (1981), $T_{b2} < 0$, the sign of $\lambda'_{T_0}(t)$ depends on the sign of $\langle P_2, \mathrm{d} \rangle = \int_0^1 P_2(x) v(x) L(x) \phi(x) dx$, which reflects the interaction of Nature dynamics with the socioeconomic factors. An approximation of this derivative with numerical estimations was provided in the previous section.

[13] We assume that each location has its own finite reserves of fossil fuels. The result can be easily modified to allow for finite global reserves which are shared by all locations. In this case, $\lambda(x,0)$ should be replaced by the global costate $\lambda(0)$.

4.1.2 The social price of fossil fuels

We turn now to the derivative $\lambda_R'(x,0)$, which is characterized in the following proposition.

Proposition 3 *Let $\bar{\xi} = -\lambda\lambda_{T_0}$ be the steady-state social price of the climate externality which is independent of heat transfer when $\hat{r} = 0$. Then the sign of $\lambda_R'(x,0)$ is opposite to the sign of $\bar{\xi}'$.*

For the proof see Appendix.

Thus when the social price of the climate externality goes down, the social price of a finite fossil fuel reserve should go up, because there is a tendency to extract more and vice versa.

4.1.3 Temperature paths and polar amplification

The identification of such a potential impact is important since our spatial model allows us to determine the characteristics of the temperature anomaly at the Poles, i.e. at $x = \pm 1$. An increase in the temperature anomaly at the Poles is related to the phenomenon of polar amplification (PA), which increases the loss of Arctic Sea ice relative to the case where PA is not present. This in turn has consequences for melting land ice and other effects. There is growing evidence suggesting a link between more rapid Arctic warming relative to the warming of the Northern hemisphere mid-latitudes when Global Mean Yearly Temperature (GMT) increases. This phenomenon has been called Arctic amplification and is expected to increase the frequency of extreme weather events (Francis and Vavrus 2014). Melting land ice associated with a potential meltdown of Greenland and West Antarctica ice sheets due to polar amplifications might cause serious global sea level rise. It is estimated that the Greenland ice sheet holds an equivalent of 7 metres of global sea level rise, while the West Antarctica ice sheet holds the potential for up to 3.5 metres of global sea level rise (see Lenton et al. 2008).[14] On the other hand, the loss of Arctic Sea ice due to Arctic amplification may generate economic benefits by making possible the exploitation of natural resources and fossil fuel reserves which are not accessible now because of the sea ice.

[14] In the discussion about tipping points, it has been stressed that the time scale of melting of the Greenland ice sheet is much longer than Arctic Sea ice melting. However the Antarctic ice sheet could melt very fast once it gets started, but it will take an increase of 5°C of surface temperature for a serious destabilization.

Thus any polar or Arctic amplification implied by welfare maximization in the context of the spatial climate model should be taken into account.

Proposition 4 *Assuming infinite fossil fuel reserves, an increase in $\hat{r}$ in the neighborhood of $\hat{r} = 0$ is associated with an increase in PA at the North Pole for the socially optimal temperature path if the increase in $\hat{r}$ reduces the social price of the climate externality. If the increase in $\hat{r}$ increases the social price of the climate externality, then there is no association with PA. The impact from an increase in $\hat{r}$ on the Equator's temperature ($x = 0$) is ambiguous.*

For proof see Appendix.

The impact of heat transport on the social price of the climate externality depends on socioeconomic as well as natural factors. Therefore PA may emerge from an optimization model as a result of specific choices like welfare weights or existing conditions, such as the distribution of population or production damages from climate change across latitudes. It should be noted that if we assume symmetry between the two hemispheres, the result of this proposition can be extended to the South Pole.

The potential generation of extra costs and benefits to mid-latitudes due to PA resulting from the optimizing model should be taken into account by fine tuning the spatial damage function. A damage function which includes damages to latitude x caused by spillovers from temperature increases at other latitudes z, e.g. melting of land ice and potential indirect effects caused by melting of sea ice, can be written as:

$$\phi\left(x; \{T_0(t) + T_2(t) P_2(z)\}_{z=0}^{z=1}\right) [T_0(t) + T_2(t) P_2(x)] . \tag{69}$$

Damages from increased melting of land ice is a flow variable rather than a stock variable, so the flow of damages should depend upon the flow of melted land ice which depends, in turn, on the volume of available ice to melt. Consider the following high-latitude belt temperature index:

$$I(T_0(t), T_2(t); z_c) \equiv \int_{z>z_c} (T_0(t) + T_2(t) P_2(z))\, dz = \tag{70}$$

$$\int_{z\in[z_c,1]} (T_0(t) + T_2(t) P_2(z))\, dz = (1 - z_c)\left[T_0(t) + \frac{T_2(t)}{2}\right] z_c (1 + z_c) .$$

Then the damage function (69) where the high-latitude temperature anomaly affects mid-latitude damages can be specified as:

$$\phi\left(I\left(T_0\left(t\right),T_2\left(t\right);z_c\right);x\right)\left[T_0\left(t\right)+T_2\left(t\right)P_2\left(x\right)\right]. \tag{71}$$

It is plausible to assume that $\phi(\cdot)$ is positive and increasing in the index $I\left(T_0\left(t\right),T_2\left(t\right);z_c\right)$ for latitudes in the set $\{z : z \leq z_c\}$. Note that $\phi(\cdot)$ might even be negative for some high latitudes because of the potential opening of new shipping lanes and the potential opening of access to previously inaccessible natural resources and fossil fuel reserves. PA effects could become substantial if warming continues, i.e. $T_0\left(t\right)$ continues to increase.

Using (71), the current value Hamiltonian (30) for the two-mode approach becomes

$$\begin{aligned}
&H = \int_{x=0}^{x=1} v\left(x\right)L\left(x\right)\left\{\ln\left[E^{\alpha}\left(1-A\right)\right]\right.\\
&\left.-\phi\left(I\left(T_0\left(t\right),T_2\left(t\right);z_c\right);x\right)\left[T_0\left(t\right)+T_2\left(t\right)P_2\left(x\right)-bA\right]\right]-\lambda_R\left(t\right)E\left(t,x\right)\right\}dx+\\
&\lambda_{T_0}\left(t\right)\left[-BT_0\left(t\right)+\lambda E\left(t\right)\right]+\\
&\lambda_{T_2}\left(t\right)\left[\left(-B-6D\right)T_2\left(t\right)-\hat{r}D\lambda_2T_0\left(t\right)T_{b2}\right].
\end{aligned}$$

PA affects the costate variables for the two temperature modes T_0, T_2 which are now modified, relative to (34)-(35) and evolve according to

$$\dot{\lambda}_{T_0} = \left(\rho+B\right)\lambda_{T_0}+6\hat{r}DT_{b2}\lambda_{T_2}+\langle vL,-\frac{\partial I}{\partial T_0}\rangle \tag{72}$$

$$\frac{\partial I}{\partial T_0} = \phi\left(I\right)+\frac{\partial\phi}{\partial I}z_c\left(1-z_c\right)\left(1+z_c\right)T_0$$

$$\dot{\lambda}_{T_2} = \left(\rho+B+6D\right)\lambda_{T_2}+\langle vL,-\frac{\partial I}{\partial T_2}\rangle \tag{73}$$

$$\frac{\partial I}{\partial T_2} = \phi\left(I\right)P_2+\frac{\partial\phi}{\partial I}z_c\left(1-z_c\right)\left(1+z_c\right)P_2T_2.$$

The impact of PA is captured by the terms

$$\langle vL,-\frac{\partial I}{\partial T_0}\rangle\ ,\ \langle vL,-\frac{\partial I}{\partial T_2}\rangle. \tag{74}$$

Although it is difficult to provide analytical results at this stage, it is clear that the PA will affect the shadow values of the two temperature modes and, through them, the social price of the climate externality and the optimal temperature path. It is worth noting that PA effects are determined by socioeconomic factors and nature dynamics. Calibration might provide a quantification of all these effects but the insight obtained is clear.

4.1.4 Growth Effects

Recent work by Moyer et al. (2014), Dietz and Stern (2015), Moore and Diaz (2015), and Hof (2015) has stressed the potentially large impacts of climate change on the growth of economic output as well as on the level of economic output especially in poorer economies (Moore and Diaz (2015), Hof (2015)). We take this effect into account as follows. In (23) we assume the $y(x,t)$ component of output is given by

$$y(x,t;T(x,t)) = y_0(x,t)\exp\left[(g_0(x) - g_1(x)T(x,t))\,t\right]\exp[-\phi(x)T(x,t)]. \tag{75}$$

Thus the temperature anomaly reduces local growth rate $g_0(x)$ by $g_1(x)T(x,t)$. The relevant Hamiltonian for optimization using the two-mode approximation is:

$$\begin{aligned} H = {} & \\ & \int_{x=0}^{x=1} \{v(x)L(x)[\alpha \ln E(x,t) + \ln(1-A) - g_1(x)[T_0(t) + T_2(t)P_2(x)]t] \\ & -\phi(x)[T_0(t) + T_2(t)P_2(x) - bA] - \lambda_R(t)E(t,x)\}\,dx + \\ & \lambda_{T_0}(t)[-BT_0(t) + \lambda E(t)] + \\ & \lambda_{T_2}(t)[(-B-6D)T_2(t) - 6\hat{r}DT_0(t)T_{b2}(t)]. \end{aligned} \tag{76}$$

The costate equations for the temperature modes are now given by

$$\dot{\lambda}_{T_0} = (\rho + B)\lambda_{T_0} + 6\hat{r}DT_{b2}\lambda_{T_2} + \langle 1, \mathrm{d}_g\rangle \tag{77}$$

$$\dot{\lambda}_{T_2} = (\rho + B + 6D)\lambda_{T_2} + \langle P_2, \mathrm{d}_g\rangle \tag{78}$$

$$\mathrm{d}_g = \phi(x)v(x)L(x) + v(x)L(x)g_1(x)t, \tag{79}$$

with forward solutions

$$\lambda_{T_0}(t) = -\int_{s=t}^{\infty} \mathrm{e}^{-(\rho+B)t}\left[6\hat{r}DT_{b2}\lambda_{T_2}(s) + \langle 1, \mathrm{d}_g\rangle\right] ds \tag{80}$$

$$\lambda_{T_2}(t) = -\int_{s=t}^{\infty} \mathrm{e}^{-(\rho+B+6D)t}\left[\langle P_2, \mathrm{d}_g\rangle\right] ds. \tag{81}$$

From (80) and (81), it can be seen that the impact of heat/moisture transport on the costate $\lambda_{T_0}(t)$ for T_0, which determines the social cost of climate change externality, is "magnified" through the channel $6\hat{r}DT_{b2}\lambda_{T_2}(s)$. The evolution of the costate $\lambda_{T_2}(s)$, which determines this impact along with the climate parameters $\hat{r}DT_{b2}$, is determined by the socioeconomic factors $\phi(x)\,v(x)\,L(x)$ and the growth effect $v(x)\,L(x)\,g_1(x)\,t$. The growth effects of climate could therefore be important in characterizing optimal paths for fossil fuel emissions and policy instruments. Their quantitative impact is undoubtedly an interesting area for further research.

5 Optimal Climate Change Policies

5.1 Fossil Fuel Taxes

The solution of the welfare maximization problem allows us to obtain some insight into the structure of optimal fuel taxes, or equivalently, optimal carbon emission taxes. A representative firm produces output using emissions or, equivalently, fossil fuels according to the production function $y(x,t)\,E(x,t)^{\alpha}$, and faces a fossil fuel tax (or carbon tax) $\tau(x,t)$.[15] Then the profit maximizing path of fossil fuel use $E(x,t)$ is determined by

$$E^0(x,t) = \arg\max_{E(x,t)} \left\{ y(x,t)\,E(x,t)^{\alpha} - \tau(x,t)\,E(x,t) \right\}, \tag{82}$$

with

$$E^0(x,t) = \left(\frac{\tau(x.t)}{\alpha y(t,x)} \right)^{\frac{1}{\alpha-1}} \quad \text{and} \tag{83}$$

$$y(x,t)\,E^0(x,t)^{\alpha} - \tau(x,t)\,E^0(x,t) = \tag{84}$$

$$(1-\alpha)\,y(t,x)^{\frac{1}{1-\alpha}}\,\tau(x,t)^{\frac{\alpha}{\alpha-1}}\,\alpha^{\frac{\alpha}{\alpha-1}}. \tag{85}$$

Consider now the problem of the social planner whose objective is to

[15] To simplify things, we assume that competitive markets exist so that output is sold at a competitive world price normalized to one, while fossil fuels are bought at a competitive world price p_F that satisfies the arbitrage condition $(\dot{p}_F(t)/p_F(t)) = r(t)$, where $r(t)$ denotes the world interest rate. Thus τ should be interpreted as including the exogenously determined fossil fuel price.

maximize

$$\int_{x=-1}^{x=1} v(x) L(x) [\ln C(x,t) - \phi(I(T_0(t), T_2(t); z_c); x)[T_0(t) + T_2(t) P_2(x)]] dx, \tag{86}$$

subject to climate and resource availability constraints, where $C(t,x)$ is per capita consumption at latitude x and time t. The planner chooses an emission tax $\tau(x,t)$ for each latitude and then the representative firm in each latitude takes this tax as parametric and determines fossil fuel use to maximize latitude payoff according to (83). Taxes collected are given by $\tau(x,t) E^0(x,t)$. In a competitive equilibrium, the lump sum transfers from the social planner back to the consumers at latitude x at date t are equal to the taxes collected at this latitude and are given by

$$Tr(x,t) = \tau(x,t) E^0(x,t) = \tau(x,t) \left(\frac{\tau(x.t)}{\alpha y(t,x)} \right)^{\frac{1}{\alpha-1}}. \tag{87}$$

Hence in equilibrium, consumption at latitude x is

$$C(t,x) = \left[y(x,t) E^0(x,t)^{\alpha} - \tau(x,t) E^0(x,t) \right] + Tr(x,t) \implies \tag{88}$$

$$C(t,x) = y(t,x) \left(\frac{\tau(x.t)}{\alpha y(t,x)} \right)^{\frac{1}{\alpha-1}} = y(t,x)^{\frac{1}{1-\alpha}} \tau(x,t)^{\frac{\alpha}{\alpha-1}} \alpha^{\frac{\alpha}{\alpha-1}}. \tag{89}$$

With consumption determined in terms of the fossil fuel tax by (89), the social planner acting as a Stackelberg leader chooses the spatiotemporal path for the fossil fuel tax $\tau(x,t)$ to maximize the integral of discounted values of optimized objectives (86), subject to climate and resource availability constraints.[16] The current value Hamiltonian function for this problem is defined as:

$$\begin{aligned} H = & \int_{x=0}^{x=1} v(x) L(x) \left\{ \ln \left[y(t,x)^{\frac{1}{1-\alpha}} \tau(x,t)^{\frac{\alpha}{\alpha-1}} \alpha^{\frac{\alpha}{\alpha-1}} \right] - \right. \\ & \left. -\phi(I(T_0(t), T_2(t); z_c); x)[T_0(t) + T_2(t) P_2(x)] - \lambda_R(t) \left(\frac{\tau(x.t)}{\alpha y(t,x)} \right)^{\frac{1}{\alpha-1}} \right\} dx + \\ & \lambda_{T_0}(t) \left[-BT_0(t) + \lambda \int_{x=0}^{x=1} \left[\left(\frac{\tau(x.t)}{\alpha y(t,x)} \right)^{\frac{1}{\alpha-1}} \right] dx \right] + \\ & \lambda_{T_2}(t) [(-B - 6D) T_2(t) - \hat{r} D \lambda_2 T_0(t) T_{b2}]. \end{aligned} \tag{90}$$

[16] To simplify the exposition, we do not consider adaptation expenses.

To provide a first insight into the optimal tax, we consider the simplest possible case where there are infinite reserves and damages are independent of the high-latitude index $I(T_0(t), T_2(t); z_c)$. In this case the optimal tax is determined as

$$\tau^*(x,t) = \arg\max_{\tau} \left\{ v(x) L(x) \ln \left[y(t,x)^{\frac{1}{1-\alpha}} \tau(x,t)^{\frac{\alpha}{\alpha-1}} \alpha^{\frac{\alpha}{\alpha-1}} \right] \right. \tag{91}$$

$$\left. +\lambda_{T_0}(t) \lambda \left(\frac{\tau(x.t)}{\alpha y(t,x)} \right)^{\frac{1}{\alpha-1}} \right\}, \tag{92}$$

which results in

$$\tau^*(x,t) = \alpha^{\alpha} (v(x) L(x))^{\alpha-1} y(x,t) (-\lambda\lambda_{T_0}(t)\lambda)^{1-\alpha}. \tag{93}$$

For the simplest case where $\hat{r} = 0$, the optimality conditions from (90) imply that at a steady state, $\lambda_{T_0} = -\frac{\langle 1, \mathrm{d} \rangle}{(\rho+B)}$ and therefore

$$\tau^*(x,t) = \Lambda(x) y(x,t) \tag{94}$$

$$\Lambda(x) = \left[\alpha^{\alpha} (v(x) L(x))^{\alpha-1} \left(\frac{\lambda \langle 1, \mathrm{d} \rangle}{(\rho+B)} \right)^{1-\alpha} \right]. \tag{95}$$

Hence although the steady-state social price of the climate externality, i.e., $-\frac{\langle 1, \mathrm{d} \rangle}{(\rho+B)}$, is independent of location, the optimal steady-state fossil fuel tax is linear in $y(x,t)$ which can be interpreted as the output-productivity component of location x. Thus there are two sources of spatial dependence for the optimal fossil fuel tax. The first is through the proportionality factor $\Lambda(x)$ of $y(t,x)$, which depends on different welfare weights and population across latitudes. The second is the output-productivity component $y(x,t)$. Note that even if welfare weights and population differences across latitudes are ignored, e.g. $v(x) L(x) = 1$, the spatial differentiation of the fossil fuel tax is introduced by spatial differences in the output-productivity component.

In the more general case in which spatial heat transport is taken into account and damages depend on the high-latitude index, i.e. we have the case $\phi(x, I)$, then, using (72) and (73), the steady state values for λ_{T_0} and λ_{T_2} are

$$\bar{\lambda}_{T_0} = -\frac{6\hat{r} D T_{b2} \bar{\lambda}_{T_2} + \langle vL, -\frac{\partial I}{\partial T_0} \rangle}{(\rho+B)}, \quad \bar{\lambda}_{T_2} = -\frac{\langle vL, -\frac{\partial I}{\partial T_2} \rangle}{(\rho+B+6D)}. \tag{96}$$

When (96) is used to determine the optimal fossil fuel tax given by (93), it is clear that the fuel tax will be adjusted both for spatial heat transport, by the term $6\hat{r}DT_{b2}\bar{\lambda}_{T_2}$, and optimal PA effects, by the terms $\frac{\partial I}{\partial T_0}$, $\frac{\partial I}{\partial T_2}$.

It should be noted that for any given distribution of welfare weights $v(x)$ and population $L(x)$, poorer latitudes, i.e., latitudes with a relatively lower output-productivity component $y(t,x)$, are taxed less per unit emissions than richer latitudes. This result should be contrasted with the result derived under the standard assumption of compensatory transfers which indicates that a unit of emissions is taxed the same no matter which latitude belt emitted it.

In the finite reserve case it can easily be seen that the optimal tax will be

$$\tau^*(x,t) = \alpha^\alpha (v(x)L(x))^{\alpha-1} y(x,t) (\lambda_R(x,t) - \lambda\lambda_{T_0}(t)\lambda)^{1-\alpha}. \quad (97)$$

Furthermore the exact impact of spatial heat transport on the the optimal fossil fuel tax is given by

$$\frac{\partial \tau^*(x,t)}{\partial \hat{r}} = \quad (98)$$
$$(1-\alpha)\alpha^\alpha (v(x)L(x))^{\alpha-1} y(x,t) (\lambda_R(x,t) - \lambda\lambda_{T_0}(t))^{-\alpha} \left(\lambda'_R(x,t) - \lambda\lambda'_{T_0}(t)\right).$$

5.2 Equilibrium Price of Permits

Chichilnisky and Sheeran (2009) have written an important book on carbon markets which stresses the "two sided coin" feature of carbon markets: (i) Efficiency objectives can be achieved by competitive equilibrium pricing on a world market of emissions permits, e.g. a uniform world market price on such markets helps prevent "carbon leakage" and other problems caused by different carbon prices/taxes in different locations, and (ii) Equity can be achieved by allocating more permits to more deserving countries.[17]

We explore their type of carbon market in our model where permits are allocated to latitudes and latitudes are treated as sovereigns. While latitudes are not countries, data on income distribution by latitude can be used to illustrate effects of allocation of permits to latitudes by income of latitudes and also to illustrate the effects of heat/moisture transport on the optimal number of emissions permits. Since uncertainty is absent, we can't address

[17] See also Chichilnisky's (2015) discussion of carbon markets.

most issues raised in the debate between carbon taxes and carbon permit markets, e.g. Weitzman (2014), or say anything about how well markets will perform in the real world (Schmalensee and Stavins 2015). Here we just take a look at the issue of implementing Pareto Optima that are desired by a welfare optimizing planner who, perhaps, assigns higher welfare weights to more "deserving" latitudes, e.g. rapidly industrializing poorer latitudes that have not emitted much in the past relative to industrialized latitudes.

Let $P(x,t)$ denote the number of emission permits allocated to latitude x at date t. We choose units so that one permit is equivalent to emissions into the atmosphere by one unit of input of fossil fuels, E.

Assume latitude x at date $t = 0$ chooses emissions that maximize the Lagrangian,

$$\int_{t=0}^{\infty} \mathrm{e}^{-\rho t} \alpha \ln E(x,t)\, dt + \mu_x \left(\int_{t=0}^{\infty} p(t) P(x,t)\, dt - \int_{t=0}^{\infty} p(t) E(x,t)\, dt \right), \tag{99}$$

where $p(t)$ is the world market price of an emissions permit at date t. The FONC of optimization imply that

$$E(x,t) = \frac{\alpha \mathrm{e}^{-\rho t}}{\mu_x p(t)}. \tag{100}$$

We assume emissions markets are working well enough and that there are no impediments or obstructions that get in the way of permits trading at a uniform price at all latitudes. The total supply of permits at each date t is given by

$$\int_{x=0}^{x=1} L(x) P(x)\, dx, \tag{101}$$

where $L(x)$ is the population of latitude x which is assumed to be constant for simplicity, and to avoid notation clutter. It is easy to generalize the treatment here to growing populations and changing populations. Total demand by latitude x at date t is given by

$$L(x) E(x,t) = L(x) \left[\frac{\alpha \mathrm{e}^{-\rho t}}{\mu_x p(t)} \right], \tag{102}$$

while global demand at date t is given by

$$\int_{x=0}^{x=1} L(x) E(x,t)\, dx = \int_{x=0}^{x=1} L(x) \left[\frac{\alpha \mathrm{e}^{-\rho t}}{\mu_x p(t)} \right] dx. \tag{103}$$

Suppose $\{E^*(x,t), x \in [0,1], t \in [0,\infty)\}$ is a desired solution by a planner, e.g. a solution to a welfare optimization problem where poor latitudes are weighted more heavily than rich latitudes, or latitudes that have emitted more relative to others in the past during industrialization are weighted less heavily by appropriately defined weights $v(x)$.

We investigate here whether the desired solution can be implemented by choosing an allocation $\{P(x,t), x \in [0,1], t \in [0,\infty)\}$ of permits and opening a world market for permits as in Chichilnisky and Sheeran (2009). We try the allocation $P(x,t) = E^*(x,t)$ for all (x,t). Equating demand and supply at date t gives us

$$\int_{x=0}^{x=1} L(x)\left[\frac{\alpha \mathrm{e}^{-\rho t}}{\mu_x p(t)}\right] dx = \int_{x=0}^{x=1} P(x,t)\, dx = \int_{x=0}^{x=1} E^*(x,t)\, dx, \tag{104}$$

recalling that the planner's optimal solution for emissions by each latitude is given by

$$E^*(x,t) = \frac{v(x) L(x)}{\lambda^*_{R_0} \mathrm{e}^{\rho t} - \lambda \lambda^*_{T_0}(t)}. \tag{105}$$

Hence,

$$p(t) = \frac{\alpha \mathrm{e}^{-\rho t} \int_{x=0}^{x=1} (L(x)/\mu_x)\, dx}{\int_{x=0}^{x=1} E^*(x,t)\, dx} = \tag{106}$$

$$\frac{\alpha\left[\lambda^*_{R_0} - \lambda\lambda^*_{T_0}(t)\mathrm{e}^{-\rho t}\right]\int_{x=0}^{x=1}(L(x)/\mu_x)\, dx}{\int_{z=0}^{z=1} v(z) L(z)\, dz}. \tag{107}$$

Since, $\lambda^*_{T_0} < 0$ for $\hat{r} = 0$, we see that in this case $p(t)$ decreases to the asymptotic value, $\lambda^*_{R_0}$ as $t \to \infty$. When "space matters", i.e. $\hat{r} > 0$, we see from (43), (44) that the forward solutions for the costate variables are

$$\lambda^*_{T_0}(t) = -\int_{s=t}^{\infty} \mathrm{e}^{-(\rho+B)t}\left[6\hat{r} D T_{b2} \lambda^*_{T_2}(s) + \langle 1, \mathrm{d}(x)\rangle\right] ds \tag{108}$$

$$\lambda^*_{T_2}(t) = -\int_{s=t}^{\infty} \mathrm{e}^{-(\rho+B+6D)t}\left[\langle P_2, \mathrm{d}(x)\rangle\right] ds \tag{109}$$

$$\mathrm{d}(x) = \phi(x) v(x) L(x) \phi(x). \tag{110}$$

Hence, in this model, spatial transport impacts the equilibrium trading price of emissions permits in the Chichilnisky/Sheeran market in a way that can be computed in closed form, once the marginal damage function $\mathrm{d}(x)$ is known.

It is of interest to calculate the equilibrium level of wealth, W_x, of each latitude under this allocation scheme. The wealth of latitude x is given by

$$W_x \equiv \int_{t=0}^{\infty} p(t)\, P(x,t)\, dt = \tag{111}$$
$$\int_{t=0}^{\infty} \left\{ \frac{\alpha\left[\lambda_{R_0}^{*} - \lambda\lambda_{T_0}^{*}(t)\, \mathrm{e}^{-\rho t}\right] \int_{x=0}^{x=1} \left(L(x)/\mu_x\right) dx}{\int_{z=0}^{z=1} v(z)\, L(z)\, dz} \right\} P(x,t)\, dt =$$
$$\int_{t=0}^{\infty} \left\{ \frac{\alpha\left[\lambda_{R_0}^{*} - \lambda\lambda_{T_0}^{*}(t)\, \mathrm{e}^{-\rho t}\right] \int_{x=0}^{x=1} \left(L(x)/\mu_x\right) dx}{\int_{z=0}^{z=1} v(z)\, L(z)\, dz} \right\} \left\{ \frac{v(x)\, L(x)}{\lambda_{R_0}^{*} \mathrm{e}^{\rho t} - \lambda\lambda_{T_0}^{*}(t)} \right\} dt =$$
$$\int_{t=0}^{\infty} \mathrm{e}^{-\rho t} \left\{ \frac{\alpha \int_{x=0}^{x=1} \left(L(x)/\mu_x\right) dx}{\int_{z=0}^{z=1} v(z)\, L(z)\, dz} \right\} \left\{ v(x)\, L(x) \right\} dt =$$
$$\left(\frac{\alpha}{\rho}\right) \left\{ \left[\int_{x=0}^{x=1} \left(L(x)/\mu_x\right) dx \right] \frac{v(x)\, L(x)}{\int_{z=0}^{z=1} v(z)\, L(z)\, dz} \right\}.$$

Thus, we see that this particular allocation scheme results in a total wealth ratio, between locations x and x'

$$\frac{W_x}{W_{x'}} = \frac{v(x)\, L(x)}{v(x')\, L(x')} \tag{112}$$

and a per capita wealth ratio,

$$\frac{w_x}{w_{x'}} = \frac{v(x)}{v(x')}. \tag{113}$$

6 Climate Externality Price and "Safety First" Utility

The results obtained above were based on the tractability advantages of the logarithmic utility function. In this section we seek to identify the impact on the social price of climate externality and the socially optimal use of fossil fuel under a more general utility function. In particular we investigate the class of utilities where marginal disutility increases very fast relative to the logarithmic utility as consumption goes towards zero.

A more general utility function results in the following welfare function:

$$\int_{t=0}^{\infty} \mathrm{e}^{-\rho t} \left[\left[\int_{x=-1}^{x=1} v(x) L(x) U \left[\frac{y E^{\alpha} \mathrm{e}^{-\phi(x) T_{total}(x,t)}}{L(x)} \right] \right] dxdt = \right. \tag{114}$$
$$\int_{0}^{\infty} \mathrm{e}^{-\rho t} \left[\left[\int_{-1}^{1} v(x) L(x) U \left[\frac{y(x,t) E(x,t)^{\alpha} e^{-\phi(x,t)[T_b(x,t)+T(x,t)]}}{L(x)} \right] \right] \right] dxdt,$$

which is maximized by choosing the optimal path $E(x,t)$, subject to the constraints imposed by Nature dynamics and fossil fuel exhaustibility. Using the two-mode approximations and the approximations of the radiating forcing term employed above, the current value Hamiltonian for the problem is:

$$\begin{aligned}
H = & \int_{x=-1}^{x=1} \left\{ v(x) L(x) U \left[\frac{y(x,t) E(x,t)^{\alpha} e^{-\phi(x)[T_0(t)+T_2(t)P_2(x)]}}{L(x)} \right] \right. \\
& \left. -\lambda_R(t) E(t,x) \right\} dx + \\
& \lambda_{T_0}(t) \left[-BT_0(t) + \lambda E(t) \right] + \\
& \lambda_{T_2}(t) \left[(-B-6D) T_2(t) - \hat{r} D \lambda_2 T_0(t) T_{b2} \right].
\end{aligned}$$

The FONC resulting from the maximum principle, after suppressing the (x,t) arguments to ease notation when necessary, are presented below. The optimal emission path $E^*(x,t)$ satisfies

$$\frac{\alpha v(x) L(x) \left[U' \left(\breve{C}(x,t) \right) \breve{C}(x,t) \right]}{E^*(x,t)} = \lambda_R(t) - \lambda \lambda_{T_0}(t) \tag{115}$$
$$\breve{C}(x,t) = y(x) E^*(x,t)^{\alpha} \mathrm{e}^{-\phi(x)[T_0(t)+T_{b0}(t)+(T_2(t)+T_{b2}(t))P_2(x)]}. \tag{116}$$

We use $\breve{C}(x,t)$ to denote the output of the economy. We assume that this output is consumed, but the consumption value has been damaged by climate damages reflected in the exponential term. The costate variables evolve according to:

$$\dot{\lambda}_{T_0} = (\rho + B) \lambda_{T_0} + \left\langle vL, \phi U' \left(\breve{C} \right) \breve{C} \right\rangle + \lambda_{T_2} 6 \hat{r} D T_{b2} \tag{117}$$
$$\dot{\lambda}_{T_2} = (\rho + B + 6D) \lambda_{T_2} + \left\langle vL, \phi P_2 U' \left(\breve{C} \right) \breve{C} \right\rangle \tag{118}$$
$$\dot{\lambda}_R(t) = \rho \lambda_R(t). \tag{119}$$

The optimality conditions for temperature dynamics, externality dynamics and the fossil fuel constraints are the same as (37)-(40).

If the heat transport is ignored, i.e. $D = 0$, the costate variables evolve according to:

$$\dot{\lambda}_{T_0} = (\rho + B)\,\lambda_{T_0} + \left\langle vL, \phi U'\left(\breve{C}\right)\breve{C}\right\rangle \tag{120}$$
$$\dot{\lambda}_{T_2} = (\rho + B + 6D)\,\lambda_{T_2} + \left\langle vL, \phi P_2 U'\left(\breve{C}\right)\breve{C}\right\rangle$$
$$\dot{\lambda}_R(t) = \rho\lambda_R(t), \tag{121}$$

with forward solutions

$$\lambda_{T_0} = -\int_{s=0}^{\infty} \mathrm{e}^{-(\rho+B)(s-t)} \left\langle v(x)\,L(x)\,, \phi(x)\,U'\left(\breve{C}\right)\breve{C}\right\rangle ds \tag{122}$$
$$\lambda_{T_2} = -\int_{s=0}^{\infty} \mathrm{e}^{-(\rho+B+6D)(s-t)} \left\langle v(x)\,L(x)\,, \phi(x)\,P_2(x)\,U'\left(\breve{C}\right)\breve{C}\right\rangle ds.$$

Conditions (117)-(118) indicate that the neat property of the log utility function obtained above is lost because of the term $U'\left(\breve{C}\right)\breve{C}$ which emerges when general utility functions are used. In order to obtain some analytical results, we consider the class of utility functions

$$U(C) = \frac{C^{1-\gamma}}{1-\gamma}, \tag{123}$$

where γ is both the coefficient of relative risk aversion and (minus) the elasticity of marginal utility with respect to consumption, while the log utility function is the special case $\gamma = 1$. For $\gamma > 1$, we call the class of utilities "safety first" because in this case, when the consumption value is damaged due to climate change, the disutility increases faster than the logarithmic utility for which $\gamma = 1$. In the same context, an increase of γ from the value of one implies an increase in the relative risk aversion. For this class of utility functions, we have $U'\left(\breve{C}\right)\breve{C} = \breve{C}^{1-\gamma}$. The main question is whether an increase in the coefficient of relative risk aversion from the value of one will have an impact on the social price of the climate externality and the socially optimal fossil fuel path.

Using (123), the optimality condition for the optimal choice of fossil fuel use becomes

$$\frac{\alpha v\left(x\right)L\left(x\right)\breve{C}\left(x,t;\hat{r},\gamma\right)^{1-\gamma}}{E\left(x,t;\hat{r},\gamma\right)}=\lambda_{R}\left(t\right)+\lambda\lambda_{T_0}\left(t\right). \tag{124}$$

Differentiating (124) with respect to γ, evaluating the derivatives at $(\hat{r},\gamma)=(0,1)$ and using $\frac{\partial\breve{C}^{1-\gamma}}{\partial\gamma}=-\ln\breve{C}$, and suppressing $(\hat{r},\gamma)$ to ease notation, we obtain

$$\alpha v\left(x\right)L\left(x\right)\left[-\frac{1}{E\left(x,t\right)^{2}}\frac{\partial E\left(x,t\right)}{\partial\gamma}-\frac{\ln\breve{C}}{E\left(x,t\right)}\right]=\frac{\partial\lambda_{R}\left(t\right)}{\partial\gamma}+\frac{\lambda\lambda_{T_0}\left(t\right)}{\partial\gamma}. \tag{125}$$

To identify the impact of increasing γ from the value $\gamma=1$ on the social price of the climate externality, we consider expansions of any endogenous variable $\zeta\left(t;\hat{r},\gamma\right)$ of our model with respect to $(\hat{r},\gamma)$ around the point $(\hat{r},\gamma)=(0,1)$, or

$$\zeta\left(t;\hat{r},\gamma\right)=\zeta\left(t;0,1\right)+\frac{\partial\zeta\left(t;0,1\right)}{\partial\hat{r}}\hat{r}+\frac{\partial\zeta\left(t;0,1\right)}{\partial\gamma}\left(\gamma-1\right)+o\left(\hat{r},\left|\gamma-1\right|\right). \tag{126}$$

Since we are interested in the social price of the climate externality and the use of fossil fuels, we consider the following expansions

$$\begin{aligned}\lambda_{T_0}\left(t;\hat{r},\gamma\right) &= \lambda_{T_0}\left(t;0,1\right)+\frac{\partial\lambda_{T_0}\left(t;0,1\right)}{\partial\hat{r}}\hat{r}+\frac{\partial\lambda_{T_0}\left(t;0,1\right)}{\partial\gamma}\left(\gamma-1\right)+ \qquad (127)\\ &\quad o\left(\hat{r},\left|\gamma-1\right|\right)\\ E\left(t;\hat{r},\gamma\right) &= E\left(t;0,1\right)+\frac{\partial E\left(t;0,1\right)}{\partial\hat{r}}\hat{r}+\frac{\partial E\left(t;0,1\right)}{\partial\gamma}\left(\gamma-1\right)+ \qquad (128)\\ &\quad o\left(\hat{r},\left|\gamma-1\right|\right),\end{aligned}$$

which approximate the climate externality price and the fossil fuel use. Using these expansions, we can state the following result.

Proposition 5 *Assuming no serious poverty at any location at any time, so that* $\ln\breve{C}\left(x,t\right)>0$ *for all* (x,t) *and* $\lambda_{R}\left(t\right)=0$ *for all* t, *then a small increase in the coefficient of relative risk aversion* γ *from* $\gamma=1$ *will reduce the social price of the climate externality* $-\frac{\partial\lambda_{T_0}(v;0,1)}{\partial\gamma}=\int_{s=v}^{\infty}\mathrm{e}^{-\rho(s-v)}\left\langle vL,\phi\ln\breve{C}\right\rangle(s)ds<0$.

For proof see Appendix.

Since in the safety-first class of utilities, climate damages in the utility function are realized through damages in the value of consumption, and recalling that $C = yE^{\alpha}\mathrm{e}^{-\phi T}$ and $U(C) = C^{1-\gamma}/(1-\gamma)$, it is reasonable to expect that an increase in γ from $\gamma = 1$ will reduce the price of the climate externality and the corresponding fuel tax when the stock of fossil fuels is assumed to be infinite. It should also be noted that the impact of the safety-first utility, as quantified by the derivative $\frac{\partial \lambda_{T_0}(v;0,1)}{\partial \gamma}$, depends on the socioeconomic factors $v(x), L(x), \phi(x)$ and the value of consumption $\breve{C}$ adjusted for climate change damages.

6.0.1 Consumption Discount Rates under Spatial Heat Transfer

The previous discussion made clear that allowing for spatial heat transfer has an impact on the social cost of the externality and the associated policy instruments. A question that emerges in this context is what the impact of heat transfer is on the discount rate used for discounting future flows of consumption costs and benefits. As is well known (e.g. Arrow et al. (2014) or Gollier (2007)) the consumption rate of interest, r_t, is defined in the context of the Ramsey rule as:

$$r_t = \rho - \frac{d}{dt}\ln\frac{\partial U(C(t))}{\partial C(t)}. \tag{129}$$

Consider the case where each location x is regarded as a "closed economy" in which case the consumption rate of interest can be a local equilibrium rate, that is,

$$r_t(x) = \rho - \frac{d}{dt}\ln\frac{\partial U(C(x,t))}{\partial C(x,t)}. \tag{130}$$

Define consumption after climate change damages have been accounted for by

$$\begin{aligned} \breve{C}(x,t) &= C(x,t)\,\mathrm{e}^{-\phi(x)\hat{D}(x,t)}, & (131)\\ \hat{D}(x,t) &= T_{b0}(t) + T_{b2}(t)P_2(x) + T_0(t) + T_2(t)P_2(x), & (132) \end{aligned}$$

and consider the utility function

$$U\left(\breve{C}(x,t)\right) = \frac{1}{1-\gamma}\left(C(x,t)\,\mathrm{e}^{-\phi(x)\hat{D}(x,t)}\right)^{1-\gamma}. \tag{133}$$

Using (130) we obtain:

$$r_t(x) = \rho + \gamma g(x,t) + (1-\gamma)\phi(x)\frac{d\hat{D}(x,t)}{dt} \tag{134}$$

$$g(x,t) = \frac{\dot{C}(x,t)}{C(x,t)}. \tag{135}$$

From (132),

$$\frac{d\hat{D}(x,t)}{dt} = \dot{T}_0(t) + \dot{T}_2(t) P_2(x), \tag{136}$$

since it is reasonable to assume that baseline temperature modes remain constant, i.e. $\dot{T}_{b0}(t) = \dot{T}_{b2}(t) = 0$. From the optimality conditions (37), (38)

$$\dot{T}_0 = -BT_0(t) + \lambda E^*(t) \tag{137}$$

$$\dot{T}_2 = -[B+6D]T_2(t) - 6\hat{r}DT_0(t)T_{b2} \tag{138}$$

where $E^*(t)$ is the optimal aggregate path for fossil fuel use determined as

$$E^*(x,t) = \frac{\alpha v(x) L(x)}{\lambda_R(t) - \lambda\lambda_{T_0}(t)}, E^*(t) = \int_0^1 E^*(x,t)\,dt. \tag{139}$$

Conditions (134)-(139) indicate that the climate change adjustment to the discount rate reduces to zero for a logarithmic utility function. But it is not zero for the most often considered values of γ between 1.5 and 3 (Dasgupta 2008). For $\gamma > 1$, the adjustment depends on the paths of the anomaly which are determined by socioeconomic conditions, i.e., $v(x), L(x), \phi(x)$ and Nature's spatial dynamics reflected in $P_2(t)$.

The impact of accounting for spatial heat transport in consumption discount rate, disregarding any impacts on the local consumption growth rate, is determined by

$$\frac{\partial}{\partial\hat{r}}\left(\frac{d\hat{D}(x,t)}{dt}\right) = \dot{T}_0'(t) + \dot{T}_2'(t) P_2(x). \tag{140}$$

In the proof of proposition 3 in the Appendix, the derivatives $\dot{T}_0'$ and $\dot{T}_2'(t)$

are explicitly calculated as

$$T_0'(t) = \lambda \int_{s=0}^{t} \mathrm{e}^{B(s-t)} E'(s)\, ds \tag{141}$$

$$T_2'(t) = -6\hat{r} D T_{b2} \int_{s=0}^{t} \mathrm{e}^{(B+6D)(s-t)} T_0(s)\, ds. \tag{142}$$

Thus if $E'(s) > 0$ for all s, accounting for spatial heat transport, that is increasing $\hat{r}$ from $\hat{r} = 0$ to $\hat{r} > 0$, will tend to reduce local consumption discount rates.

If we consider the case where arbitrage will force local rates $r_t(x)$ to a global equilibrium rate r_t, the spatially average consumption rate of interest will be defined as

$$r_t = \rho - \frac{d}{dt} \ln \left\{ \int_{x=0}^{x=1} \left[C(x,t)^{-\gamma} \mathrm{e}^{-\phi(x)\hat{D}(x,t)} \right] dx \right\}. \tag{143}$$

7 Concluding Remarks and Suggestions for Future Research

This paper is, to our knowledge, the first paper in climate economics to consider the combination of spatial heat transport and polar amplification. We simplified the problem by stratifying the Earth into latitude belts and assuming as in North et al. (1981) and Wu and North (2007) that the two hemispheres were symmetric so that solutions of the climate dynamics could be expanded into an infinite series of even numbered Legendre polynomials.

In order to obtain analytical tractability of the climate dynamics across latitude belts and to solve the economic infinite horizon welfare economics problem, we introduced some approximations to the climate dynamics and some specializations to specific utility functions.

First we follow recent research suggesting that global mean warming is linearly proportional to cumulative carbon emissions (e.g Matthews at al. 2009, Pierrehumbert 2012-2013, 2014). Second we truncated the Legendre polynomial expansion of the climate dynamics to a small number of modes. Third, we built upon work by Alexeev et al. (2005), Langen and Alexeev (2007), and Alexeev and Jackson (2012) to motivate our specification of the heat transport function across latitudes as a function of global average temperature. This specification imparts a nonlinearity which we approximated by series expansion around the case of no polar amplification where heat transport is linear.

In this paper we use logarithmic utility and exponential specification of climate damages as a function of temperature, except in section 6 where we use a more general utility function. We analyzed spillover effects from higher latitudes onto lower latitudes because of amplification of warming on the higher latitudes. Our main contributions are the following.

First, we showed that it is possible to build climate economic models that include the very real climatic phenomena of heat transport and high latitude amplification of warming (i.e. "polar amplification") and still maintain analytical tractability. Since analytical tractability is essential for understanding the output of more complicated and realistic models, we view this as an important - maybe the most important - contribution of this line of research. It is interesting to note the importance of the work by North and others in showing how models with spatial transport in climate dynamics can be made analytically tractable by use of the "right" mathematics, e.g. bases of even number Legendre polynomials and spherical harmonics. This kind of work is used heavily to understand the computational output of much more complicated and realistic climate models. We consider our work as initiating a similar line of research for the joint modeling of coupled climate dynamics and economic dynamics. We believe that our finding regarding the link between heat transfer, polar amplification, optimal fuel taxes, and permits' markets illustrates the importance of directing future research in climate change economics towards addressing the impact of spatial energy transport across the globe.

Second, we showed that the optimal tax function, i.e. the marginal social cost of emissions, depended upon the distribution not only of welfare weights but also population across latitudes, the distribution of marginal damages across latitudes and cross latitude interactions of marginal damages, along with Nature dynamics. These dynamics are reflected in the decomposition of the temperature field into modes via the expansion of the climate dynamics into a series of even numbered Legendre polynomials. The formulas we obtained are quite interpretable and comparative dynamics can be quite easily done on their components.

Third, we derived and compared optimal solutions under (i) no heat transport, (ii) heat transport but no polar amplification, and (iii) both heat transport and polar amplification.

Fourth, we compared the solution for optimal taxes under the standard assumption of compensatory transfers, so that a unit of emissions is taxed the same no matter which latitude belt emitted it, with the solution for optimal taxes in which there are no compensatory transfers at all. In this latter case the poorer latitudes are taxed less per unit emissions than richer latitudes. While this is obvious for the direction of the tax, we give a formula that shows both how the interaction of the climate system with the economic system feeds into a formula for the optimal tax per unit emissions, and the way in which optimal taxes are differentiated across locations. We also discuss the possibility that an increase in the heat transfer towards the Poles may increase or reduce fossil fuels taxes. This is an important observation because it provides a direct link between spatial heat transport in climate dynamics, which are usually disregarded in IAMs at the analytically tractable level in the simplicity hierarchy, and optimal economic policy. In the context of policy analysis, we analyze the spatial transport impacts on the equilibrium trading price of emissions permits in the Chichilnisky/Sheeran market and we calculate the equilibrium level of wealth of each latitude under this allocation scheme.

Fifth, by using a more general utility function, we showed that an increase in the coefficient of relative risk aversion from the value of one will reduce the social price of the climate externality. The more general utility function also allowed us to characterize the impact of spatial heat/moisture transfer on the discount rate which is appropriate for discounting future consumption costs and benefits.

Future research could move in different directions. The most important is that extensive computational work should be done to locate sufficient conditions for spatial heat transport and polar amplification to quantitatively matter significantly for welfare economics at different locations on the planet. We believe the ideal would be to conduct computational work like that of Cai et al. (2015) to assess the quantitative importance of taking into account heat transport. In addition, it would be valuable to extend the results in this paper to two-dimensional space where heat transport occurs across both latitude and longitude. Brock et al. (2013) did this for the case of linear heat transport but did not include polar amplification. Another area of future research would be to extend our current paper and the Desmet and Rossi-Hansberg (2015) paper, which addresses migration responses to climate change, to include the impact of heat and moisture transport across the globe. This research could build on the work of Desmet and Rossi-Hansberg (2010, 2015), and Boucekkine et al. (2009, 2013). Moreover, we have ignored the linkage between the dynamics of the carbon cycle and temperature dynamics in order to focus sharply on the additional impact of spatial heat transport on the temperature dynamics. Future research is needed to model the interaction of spatial heat transport with the carbon cycle and land use changes.

We conclude this paper by hoping that our analytical results have helped make the case that a serious dynamic climate science phenomenon like spatial heat transport can be included in analytically tractable simple climate economics models. We believe that the results in this paper suggest that spatial heat transfer and polar amplification could have a potentially important impact on climate change policy.

8 Appendix

Proof of Proposition 2

Take the derivative of (43) with respect to $\hat{r}$, evaluated at $\hat{r} = 0$, to obtain

$$\dot{\lambda}'_{T_0}(t) = (\rho + B)\,\lambda'_{T_0}(t) + 6DT_{b2}(t)\,\lambda_{T_2}(t)\,. \tag{144}$$

The forward solution for λ'_{T_0}, assuming $T_{b2}(t) = T_{b2}$ constant across dates, is

$$\lambda'_{T_0}(t) = -6DT_{b2}\int_{q=t}^{\infty} \mathrm{e}^{-(\rho+B)(q-t)}\lambda_{T_2}(q)\,dq. \tag{145}$$

Using (44) the forward solution for $\lambda_{T_2}(q)$ is

$$\lambda_{T_2}(q) = -\langle P_2, \mathrm{d}\rangle \int_{s=q}^{\infty} \mathrm{e}^{-(\rho+B+6D)(s-q)}ds. \tag{146}$$

Then

$$\lambda'_{T_0}(t) = (6DT_{b2})\,\langle P_2, \mathrm{d}\rangle \int_{q=t}^{\infty} \mathrm{e}^{-[2(\rho+B)+6D](q-t)}dq, \tag{147}$$

which implies that

$$\lambda'_{T_0}(t) = \frac{(6DT_{b2})\,\langle P_2, \mathrm{d}\rangle}{2(\rho+B)+6D}\mathrm{e}^{-[2(\rho+B)+6D]t}. \tag{148}$$

□

Proof of Proposition 3

Recall the optimality condition for the optimal emission path

$$E(x,t) = \frac{\alpha v(x) L(x)}{\lambda_R(x,t) + \xi(t)} \quad , \quad \xi(t) = \lambda \lambda_{T_0}(t) \,. \tag{149}$$

Combining this condition with the constraint of finite fossil fuel reserves in each location, we obtain

$$\alpha v(x) L(x) \int_{s=0}^{\infty} \left[\frac{1}{\lambda_R(x,0) \mathrm{e}^{\rho t} + \xi(s)} \right] ds = R_0(x) \,. \tag{150}$$

We evaluate the last integral at $\hat{r} = 0$, where $\xi(s) = \bar{\xi}$ constant and solve for $\lambda_R(x,0)$ to obtain:

$$\lambda_R(x,0) = \bar{\xi} / \left\{ \exp\left[\frac{\rho \bar{\xi} R_0(x)}{\alpha v(x) L(x)} \right] - 1 \right\}, \tag{151}$$

since

$$\int_{s=0}^{\infty} \left[\frac{1}{\lambda_R(x,0) \mathrm{e}^{\rho s} + \bar{\xi}} \right] ds = \frac{1}{\rho \bar{\xi}} \left[\ln\left(\frac{\lambda_R(x,0) + \bar{\xi}}{\lambda_R(x,0)} \right) \right]. \tag{152}$$

Differentiating (149) and (150) with respect to $\hat{r}$, we obtain

$$E'(x,t) = -\alpha v(x) L(x) \frac{\lambda_R'(x,0) \mathrm{e}^{\rho s} + \xi'(t)}{\left[\lambda_R(x,0) \mathrm{e}^{\rho s} + \bar{\xi} \right]^2} \tag{153}$$

$$-\alpha v(x) L(x) \int_{s=0}^{\infty} \left(\frac{\lambda_R'(x,0) \mathrm{e}^{\rho s} + \xi'(t)}{\left[\lambda_R(x,0) \mathrm{e}^{\rho s} + \bar{\xi} \right]^2} \right) ds = 0. \tag{154}$$

Multiplying the nominator and denominator of the integral in (154) by $\mathrm{e}^{-\rho s}$

and solving for $\lambda_R'(x,0)$, we obtain:

$$\lambda_R'(x,0) = \tag{155}$$

$$\frac{-\int_{s=0}^{\infty}\xi'(s)\left\{1/\left[\lambda_R(x,0)\,e^{\rho s}+\bar{\xi}\right]^2\right\}ds}{\int_{s=0}^{\infty}\left\{e^{\rho s}/\left[\lambda_R(x,0)\,e^{\rho s}+\bar{\xi}\right]^2\right\}ds} = \tag{156}$$

$$\frac{-\bar{\xi}'\int_{s=0}^{\infty}\left\{1/\left[\lambda_R(x,0)\,e^{\rho s}+\bar{\xi}\right]^2\right\}ds}{\int_{s=0}^{\infty}\left\{e^{\rho s}/\left[\lambda_R(x,0)\,e^{\rho s}+\bar{\xi}\right]^2\right\}ds} = \tag{157}$$

$$-\bar{\xi}'\left\{\rho\lambda_R(x,0)\left[\lambda_R(x,0)+\bar{\xi}\right]\right\}\int_{s=0}^{\infty}\left\{\frac{1}{\left[\lambda_R(x,0)\,e^{\rho s}+\bar{\xi}\right]^2}\right\}ds, \tag{158}$$

since

$$\int_{s=0}^{\infty}\left\{e^{\rho s}/\left[\lambda_R(x,0)\,e^{\rho s}+\bar{\xi}\right]^2\right\}ds = \frac{1}{\rho\lambda_R(x,0)\left[\lambda_R(x,0)+\bar{\xi}\right]}. \tag{159}$$

It follows from (158) that $\lambda_R'(x,0)$ and $\bar{\xi}'$ have opposite signs.□

Proof of Proposition 4

The impact of heat transport on the optimal temperature paths requires the computation of the derivative of $T(x,t)$ with respect to $\hat{r}$ which, using the two-mode approach, is defined as:

$$T'(x,t) = T_0'(t) + T_2'(t)\,P_2(x). \tag{160}$$

Differentiating the optimality conditions for the state variables we obtain:

$$\dot{T}_0'(t) = -BT_0'(t) + \lambda E'(t)\ \ ,\ T_0'(0) = 0 \tag{161}$$

$$\dot{T}_2'(t) = -(B+6D)\,T_2' - 6\hat{r}DT_0(t)\,T_{b2}\ ,\ T_2'(0) = 0 \tag{162}$$

$$E'(x,t) = \frac{-\alpha v(x)\,L(x)\left(\lambda_R'(x,0)\,e^{\rho t} - \lambda\lambda_{T_0}'(t)\right)}{\left(\lambda_R(x,0)\,e^{\rho t} - \lambda\lambda_{T_0}(t)\right)^2}. \tag{163}$$

We evaluate ((161) at $T_{b2} < 0$, the solution of is:

$$T_0'(t) = e^{-Bt}\left(T_0'(0) + \lambda\int_{s=0}^{t} e^{Bs}E'(s)\,ds\right) = \lambda\int_{s=0}^{t} e^{B(s-t)}E'(s)\,ds \tag{164}$$

while the solution of (162), evaluated at $T_{b2} < 0$, is

$$T_2'(t) = \mathrm{e}^{-(B+6D)t}\left(T_2'(0) - 6\hat{r}DT_{b2}\int_{s=0}^{t}\mathrm{e}^{(B+6D)s}T_0(s)\,ds\right) \quad (165)$$

$$= -6\hat{r}DT_{b2}\int_{s=0}^{t}\mathrm{e}^{(B+6D)(s-t)}T_0(s)\,ds > 0. \quad (166)$$

Assume that the fossil fuel reserves are infinite so that $\lambda_R(x,0) = 0$ for all t. The derivative $E'(s)|_{s=0}^{t} = \left[\alpha vL\lambda\lambda_{T_0}'(t)\right]/\left(\lambda\lambda_{T_0}(t)\right)^2$ could be either positive or negative, depending on the sign of the derivative of the social price of the externality λ_{T_0}', which is given in Proposition 2. Assume that socioeconomic and natural factors are such that $\lambda_{T_0}'(t) > 0$, $(\lambda_{T_0}(t) < 0)$, then $E'(s) > 0$ at all locations x and at all dates s when reserves are infinite. In this case $T_0'(t) > 0$. Recall that $T_0(t)$ is global average temperature at date t. Hence we should expect global average temperature to increase when more fossil fuels are used. Solving (162) and using $T_0(t) > 0$, $T_{b2} < 0$, we obtain $T_2'(t) > 0$ for all $t > 0$, or

$$T_2'(t) = \mathrm{e}^{-(B+6D)t}\left(T_2'(0) - 6D\int_{s=0}^{t}\mathrm{e}^{(B+6D)s}T_0(s)\,T_{b2}(s)\,ds\right)$$

$$= -\int_{s=0}^{t}\mathrm{e}^{(B+6D)(s-t)}T_0(s)\,T_{b2}(s)\,ds > 0. \quad (167)$$

Then from the derivative (160) we obtain:

$$T'(x,t) = T_0'(t) + T_2'(t)\,P_2(x) = T_0'(t) + T_2'(t)\left[\frac{1}{2}\left(3x^2 - 1\right)\right] \quad (168)$$

$$T'(0,t) = T_0'(t) + T_2'(t)\,P_2(0) = T_0'(t) - T_2'(t)\left(\frac{1}{2}\right) \quad (169)$$

$$T'(1,t) = T_0'(t) + T_2'(t)\,P_2(1) = T_0'(t) + T_2'(t) > 0, \quad (170)$$

i.e. temperature may fall or even rise at the Equator and rises at the North Pole. Hence we obtain PA when $\hat{r}$ increases from $\hat{r} = 0$ in the case where reserves are infinite at all locations.

If $E'(s) < 0$, the signs of inequalities are reversed. That is, $T_0'(t) < 0, T_2'(t) < 0$ and

$$T'(1,t) = T_0'(t) + T_2'(t)\,P_2(1) = T_0'(t) + T_2'(t) < 0. \quad (171)$$

In this case temperature may fall or even rise at the Equator and fall at the North Pole.□

Proof of Proposition 5

We differentiate the dynamical system (120)-(121) with respect to γ, using the utility function $U(C) = \frac{C^{1-\gamma}}{1-\gamma}$, to obtain at $\gamma = 1$:

$$\dot{\lambda}_{T_0} = (\rho + B)\lambda_{T_0} + \left\langle vL, \phi U'\left(\breve{C}\right)\breve{C}\right\rangle + \lambda_{T_2} 6\hat{r} D T_{b2} \tag{172}$$

$$\dot{\lambda}_{T_2} = (\rho + B + 6D)\lambda_{T_2} + \left\langle vL, \phi P_2 U'\left(\breve{C}\right)\breve{C}\right\rangle$$

$$\dot{\lambda}_R(t) = \rho\lambda_R(t). \tag{173}$$

$$\frac{\partial\dot{\lambda}_{T_0}}{\partial\gamma} = (\rho + B)\frac{\partial\lambda_{T_0}}{\partial\gamma} + \left\langle vL, -\phi\ln\breve{C}\right\rangle + 6\hat{r} D T_{b2}\frac{\partial\lambda_{T_2}}{\partial\gamma} \tag{174}$$

$$\frac{\partial\dot{\lambda}_{T_2}}{\partial\gamma} = (\rho + B + 6D)\frac{\partial\lambda_{T_2}}{\partial\gamma} + \left\langle vL, -\phi P_2\ln\breve{C}\right\rangle \tag{175}$$

$$\frac{\partial\dot{\lambda}_R(t)}{\partial\gamma} = \rho\frac{\partial\lambda_R}{\partial\gamma}. \tag{176}$$

The quantity $\ln\breve{C}$ can be computed at $(\hat{r}, \gamma) = (0, 1)$ as

$$\ln\breve{C}(x, t; 0, 1) = \ln y(x, t) + \alpha\ln E(x, t; 0, 1) - \tag{177}$$

$$\phi(x)\left[T_0(t; 0, 1) + T_{b0}(t) + \left(T_2(t; 0, 1) + T_{b2}(t)\right)P_2(x)\right]. \tag{178}$$

It is natural to put $T_{b0}(t) = \bar{T}_{b0}$, $T_{b2}(t) = \bar{T}_{b2}$ at steady-state values for all t because the climate system without humans would plausibly be at the steady state. Making the no-serious-poverty assumption at any location at any time, so that $\ln\breve{C}(x, t) > 0$ for all (x, t), we can compute the forward solution for $\frac{\partial\lambda_{T_0}}{\partial\gamma}$ at $\hat{r} = 0$ from (174). Thus we have for the forward solution and the steady-state value of $\frac{\partial\lambda_{T_0}(v;0,1)}{\partial\gamma}$:

$$\frac{\partial\lambda_{T_0}(v; 0, 1)}{\partial\gamma} = \int_{s=v}^{\infty} \mathrm{e}^{-(\rho+B)(s-v)}\left\langle vL, \phi\ln\breve{C}\right\rangle(s)ds > 0 \tag{179}$$

$$\left(\overline{\frac{\partial\lambda_{T_0}(0, 1)}{\partial\gamma}}\right) = \bar{\lambda}_{T_0\gamma} = \frac{\left\langle vL, \phi\ln\breve{C}\right\rangle}{(\rho + B)} > 0. \tag{180}$$

□

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