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Diffusion-Induced Instability and Pattern Formation in Infinite Horizon Recursive Optimal Control

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Abstract

This paper develops local stability analysis for deterministic optimal control theory for recursive infinite horizon intertemporal optimization problems where there is a continuum of spatial sites and the state variable can diffuse over these sites. We identify sufficient conditions for a type of local instability which emerges from the interaction of the discount rate on the future, the curvature of the Hamiltonian, and the spatial features of the problem. We call this phenomenon "diffusion-induced instability". We give a self-contained treatment of the control theory needed for our particular context because, although it is covered in the mathematical literature, our treatment is much more transparent for economists readers. We illustrate our analytical methods with three stylized applications. The first application is the optimal management of spatially connected human dominated ecosystems. The second and third applications are harvesting of spatially interconnected renewable resources.

JEL Classification C6, Q2

Keywords: Spatial analysis, Pattern formation, Optimal Control, Diffusion-Induced Instability.

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1 Introduction

This paper develops deterministic optimal control theory for recursive intertemporal optimization problems where there is a continuum of spatial sites and the state variable can diffuse over these sites. We develop a set of sufficient conditions on the Hamiltonian for the optimally controlled system to be locally asymptotically stable. We also develop a set of sufficient conditions on the Hamiltonian for the spatial diffusion to cause local instability of optimal steady states. This contrasts with intuition because we would expect diffusion to contribute to stability, not to instability. We show that if the discount rate on the future is small enough relative to the Hamiltonian curvature (the product of the absolute values of the second derivatives of the Hamiltonian with respect to the state variable and to the co-state variable), the usual value loss turnpike global asymptotic stability result is restored. This is so because if the discount rate on the future is zero, the optimally controlled system minimizes long-run value loss relative to the optimal steady state. The optimal steady state solves a strictly concave optimization problem over a convex set when the discount rate is zero. Intuition suggests that the same type of result should hold if the discount rate on the future is close enough to zero provided that there is strict Hamiltonian curvature present. This intuition can be made rigorous, as we show below. Diffusion-induced instability is more difficult for intuition to grasp. It turns out that diffusion-induced instability can only occur when the Hamiltonian curvature is large enough relative to the discount rate.

We believe that the new contribution of this paper lies in the development of the theory of spatial diffusion-induced stability and instability to infinite horizon recursive optimal control problems of the type that lie at the very foundation of dynamic economic theory. The literature on spatial diffusion-induced stability and instability in dynamical systems is huge. But the literature on optimally controlled spatially connected systems is much smaller. The literature on stability analysis in recursive optimal control problems with infinite dimensional state spaces is even smaller yet.[1] We believe the analysis of diffusion-induced instability developed here is new.

In order to develop this theory we first present in this paper a self contained treatment of the usual Hamiltonian formalism for our spatial context. After that task is done we present our stability analysis. Then we apply our analysis to three stylized problems of dynamic economics. The paper ends with a conclusions Section.

[1] See for example Carlson, Haurie and Leizarowitz (1991).

2 Optimal Control in Space-Time: An Extension of Pontryagin's Principle

In this Section we present necessary and sufficient conditions for the optimal control of systems in space-time. We need this basic material in order to define precisely the notion of "Hamiltonian" that we will use to develop the stability results. We will use this material to study the emergence of spatial heterogeneity and pattern formation through diffusion-driven instability in an infinite time horizon. After stability conditions are explored, we will develop two stylized applications: (i) An optimal ecosystem management model where the ecosystems are spatially connected; and (ii) Two renewable resource harvesting models where the renewable resource itself diffuses across space.

We start by considering an optimal control problem defined in the spatial domain $z \in \mathcal{Z} = [z_0, z_1]$ and the time domain $t \in [0, \infty)$. Let $x(t,z), u(t,z)$ be the scalar state and control variables respectively at time t and spatial point z, taking values in compact sets $\mathcal{X}$ and $\mathcal{U}$. Let $f(x(t,z), u(t,z))$ be a net benefit function satisfying standard concavity assumptions and consider the following infinite horizon optimal control problem:

$$\max_{\{u(t,z)\}} \int_{z_0}^{z_1} \int_0^{\infty} e^{-\rho t} f(x(t,z), u(t,z))\, dtdz \tag{1}$$

$$\text{s.t. } \frac{\partial x(t,z)}{\partial t} = g(x(t,z), u(t,z)) + D\frac{\partial^2 x(t,z)}{\partial z^2}, \; x(t_0, z) \text{ given} \tag{2}$$

with possible spatial boundary conditions

$$(i)\; x(t,z_0) = x(t,z_1) = \bar{x}(t), \; \frac{\partial x(t,z_0)}{\partial z} = \frac{\partial x(t,z_1)}{\partial z}, \; \forall t, \tag{3}$$

the space is a circle, and 'smooth pasting'

$$(ii)\; \left.\frac{\partial x(t,z)}{\partial z}\right|_{z=z_0} = \left.\frac{\partial x(t,z)}{\partial z}\right|_{z=z_1} = 0, \; \forall t: \tag{4}$$

zero flux on the boundaries of the spatial domain

$$(iii)\; x(t,z_0) = x(t,z_1) = 0, \; \forall t: \text{ hostile boundary} \tag{5}$$

In the above problem the transition equation (2) states that the rate of change of the state variable, which could be, for example, the concentration of an environmental stock (*e.g.,* greenhouse gasses, phosphorus in a lake), or the concentration of a biological resource, at a given spatial point is determined by a general growth function $g(x(t,z), u(t,z))$, which reflects the kinetics of the state variable, and by dispersion reflected by $D\frac{\partial^2 x(t,z)}{\partial z^2}$. Here the coefficient, $D > 0$, is the diffusion coefficient. We use the basic assumptions regarding diffusion, which are those of the classical approach, *i.e.*, the flux of the stock is assumed to be proportional to the gradient of the size of the stock. In a resource application the stock variable is resource concentration. We assume the movement of the stock is from high concentration towards low concentration. Condition (3) which we labelled "circle" implies that the spatial domain is a circle of given length L. In this case the set $[z_0, z_1] = [0, L]$. The smooth pasting condition, $\frac{\partial x(t,z_0)}{\partial z} = \frac{\partial x(t,z_1)}{\partial z}$, says that the gradient of the resource is the same at the beginning and at the end of the circle. This is so because the beginning and the end refer to the same point. Condition (4), which we labeled "zero flux", is a zero flux condition. That is, it is assumed that there is no external input on the boundary of the spatial domain.[2] Condition (5), which we labelled "hostile boundary", is an alternative boundary condition indicating that the exterior of the spatial domain is completely hostile to the resource. For example if x denotes a species biomass, "hostile boundary" implies that individuals that cross the boundary die.[3] It is well-known that hostile boundary conditions can cause heterogeneity in steady states across space simply because heterogeneity is "forced" by the boundary conditions at the end points of the spatial domain. Much of what we do below is carried out under circle boundary conditions that allow homogeneity in steady states because we want to study how optimization itself can generate heterogeneity in spatial steady states.

Problem (1) is an optimal control problem in a fixed spatial domain and an infinite time horizon. The circle and hostile boundary conditions (5) can be associated with a type of a "fixed endpoint problem in the spatial domain" for the state variable, since the terminal value of the state variable is fixed at terminal space. The zero flux terminal condition (4) corresponds to a "free endpoint problem in the spatial domain" for the state variable, since the terminal value of the state variable is not *a priori* specified at terminal space. These terminal conditions in the space domain will be used to specify the appropriate transversality conditions for the problem.

Problem (1) to (5) has been analyzed in more general forms (*e.g.* Lions, 1971).[4] However, we give a self contained treatment of Pontryagin's principle for this problem, because it is in the spirit of the optimal control formalism

[2] The zero flux boundary condition is usually imposed so that the organizing pattern is self-organizing and not driven by boundary conditions (Murray 2003, Vol II, p. 82).

[3] See for example, Murray (2003, Vol II, p. 1200 or Neubert (2003).

[4] Similar conditions have been derived for other cases, such as the control of parabolic equations (Raymond and Zidani,1998, 1999), boundary control (Lenhart, Liang and Protopeskou, 1999), or distributed parameter control (Carlson et al., 1991; Lenhart and Bhat, 1992; Bhat et.al., 1999; Bhat, Fisher and Lenhart, 1999).

used by economists, and thus can be used for other applications as well as the applications given here. Furthermore, as we shall see for recursive contexts treated here, the use of Pontryagin's principle in continuous time and in space allows for a drastic reduction in the dimensionality of the dynamic system describing the phenomenon and makes the problem tractable. Our results are presented below, with proofs in the Appendix.

Maximum Principle under Diffusion: Necessary Conditions (MPD-NC) *Let $u^* = u^*(t,z)$ be a choice of instrument that solves problem (1) to (5) and let $x^* = x^*(t,z)$ be the associated path for the state variable. Then there exists a function $q(t,z)$ such that for each t and z :*

1. $u^* = u^*(t,z)$ *maximizes the generalized current value Hamiltonian function*

$$\tilde{H}(x(t,z), u, q(t,z)) = f(x,u) + q(t,z)\left[g(x(t,z), u(t,z)) + D\frac{\partial^2 x(t,z)}{\partial z^2}\right] \quad (6)$$

or for interior solutions:

$$f_u + q(t,z)\, g_u = 0 \quad (7)$$

2.

$$\frac{\partial q(t,z)}{\partial t} = \rho q(t,z) - \frac{\partial H}{\partial x} - D\frac{\partial^2 q(t,z)}{\partial z^2} = \rho q(t,z) - \left(f_x + q(t,z)\, g_x + D\frac{\partial^2 q(t,z)}{\partial z^2}\right) \quad (8)$$

$$\frac{\partial x(t,z)}{\partial t} = g(x(t,z), u^*(t,z)) + D\frac{\partial^2 x(t,z)}{\partial z^2} \quad (9)$$

evaluated at $u^* = u^*(x(t,z), q(t,z))$, $H = f(x,u) + qg(x,u)$

3. *The following limiting intertemporal transversality condition holds*[5]

$$\lim_{T\to\infty} e^{-\rho T} \int_{z_0}^{z_1} q(T,z)\, x(T,z)\, dz = 0 \quad (10)$$

4. *The following spatial transversality conditions hold for all dates t:*

[5] We state here a general form for the transversality conditions. These conditions are modified and supplemented appropriately for each boundary specification in the Appendix.

- *For the circle:*

$$q(t, z_0) = q(t, z_1), \ \frac{\partial q(t, z_1)}{\partial z} = \frac{\partial q(t, z_0)}{\partial z} \tag{11}$$

- *For zero flux:*

$$\frac{\partial q(t, z_1)}{\partial z} = \frac{\partial q(t, z_0)}{\partial z} = 0 \tag{12}$$

- *For hostile boundary:*

$$q(t, z_0) = q(t, z_1) = 0 \tag{13}$$

Conditions (7)-(9) along with the transversality conditions can be used to characterize the optimized dynamic system in continuous time space. It is interesting to note that (8)-(9) is a Modified Hamiltonian Dynamic System (MHDS) defined in continuous space-time. In this system the diffusion coefficient for the costate variable is negative, and it is the opposite of the state variable's diffusion coefficient. Since the costate variable can be interpreted as the shadow value of the state variable, negative diffusion implies that the movement in space is from low shadow values to higher shadow values. This makes economic sense because the costate variable is a price and the state variable diffuses from high concentration to low concentration. Therefore one would expect the price of the state variable to move in the opposite direction. Furthermore, the opposite signs of the diffusion coefficient for the state and the costate variable imply that time 'runs backward' in the state variable and 'runs forward' in the costate variable. The costate variable is equal to the capitalized value of the stream of future value flows (appropriately interpreted). That is, it is a forward capitalization type variable in capital theoretic terms.[6]

The conditions derived above are essentially necessary conditions, although it can be shown that under concavity assumptions they are also sufficient. Sufficiency conditions can also be derived by extending Arrow and Kurz (1970) type sufficiency conditions of optimal control. We state and prove a sufficiency result in the Appendix.

[6] It should also be noted that because the diffusion coefficient D is a constant independent of the value of the state variable across sites, the optimal control $u^*(t, z)$ depends only on the values of the state and costate variables of its "own site". All dependencies of u^* on other sites is summarized in $q(t, z)$.

3 Pattern Formation in Optimally Controlled Systems

Conditions (7)-(13) characterize the optimal paths for the state and control variables and the associated path for the costate variable in space-time. An important question in this context is whether these optimal paths are characterized by spatial heterogeneity. Spatial heterogeneity occurs when the values of the state, costate and control variables are different at different spatial points at the same time. Spatial homogeneity occurs when the values of these variables do not change across spatial points. We wish to study the forces that lead to spatial heterogeneity which persists with the passage of time, because this could lead to formation of stable spatial patterns.

We are going to approach this problem by examining how diffusion, regarded as a perturbation, affects the steady state of an optimal control problem without spatial considerations. This is the special case of problem (1)-(2) with $D = 0$. From optimal control theory, with $D = 0$, we know that under appropriate concavity assumptions, if a steady state defined as (x^*, q^*) such that $\dot{x} = 0$, $\dot{q} = 0$ exists, then this steady state will have the local saddle point property or it will be unstable (*e.g.* Kurz, 1968; Levhari and Liviatan, 1972). Saddle point stability is a concept of conditional stability and implies that for the general n-dimensional problem there exists an n - dimensional *locally* stable manifold such that if the state-costate dynamical system[7] starts on this manifold, in the neighborhood of the steady state, it remains on it for all time and converges to the saddle point steady state. Brock and Scheinkman (1976), Cass and Shell (1976), and Rockafellar (1976) extended this local stability concept to global asymptotic stability (GAS) by introducing a *curvature assumption.* This curvature assumption is a sufficient condition which is stated in terms of the Hamiltonian of the control problem. The term, GAS, as used here means that the solution manifold of the optimal control problem is globally stable, in the sense that for any initial conditions on the solution manifold, the state-costate trajectories remain on the solution manifold and converge to the saddle point steady state.

By definition when $D = 0$ the steady state with the saddle point property is spatially homogeneous or a Flat Optimal Steady State (FOSS). If diffusion destabilizes the stable manifold of this steady state, then the result might be the emergence of a regular stable patterned distribution for the state, costate and control variables across the spatial domain. This is basically the idea behind the Turing mechanism (Turing, 1952) as discussed in Murray (2003) for

[7] In infinite time horizon optimal control terminology, this is the Modified Hamiltonian Dynamic System (MHDS).

diffusion-driven instability and pattern formation. The mechanism suggests that an asymptotically stable, in the absence of diffusion, spatially homogeneous steady state, can be destabilized locally by perturbations induced by diffusion. This mechanism has been extensively used to study pattern formation in biological applications,[8] which do not include optimal control considerations. We believe our current paper is the first time that the mechanism of diffusion-induced instability has been used to study pattern formation in optimal control problems.

In order to use the mechanism of diffusion-induced instability for studying the local stability of the flat steady state (x^*, q^*), we follow standard methods and use the linearization of the dynamical system defined by (7)-(9). Defining the maximized current value Hamiltonian or pre-Hamiltonian for the flat system by

$$H^0(x,q) = \max_u \{f(x,u) + q(t)[g(x(t),u(t))]\} \tag{14}$$

the Jacobian of the MHDS at the flat steady state (x^*, q^*) is defined as

$$J^0(x^*,q^*) = \begin{bmatrix} H^0_{xq}(x^*,q^*) & H^0_{qq}(x^*,q^*) \\ -H^0_{xx}(x^*,q^*) & \rho - H^0_{qx}(x^*,q^*) \end{bmatrix} \tag{15}$$

As shown by Kurz (1968), the eigenvalues of $J^0(x^*,q^*)$, for $\rho > 0$, either have positive real parts or they have opposite signs.[9] No eigenvalue with zero real parts exists. Assume that the flat steady state (x^*, q^*) has the saddle point property. In order to examine whether this state can be locally destabilized by diffusion, we need to find sufficient conditions for the negative eigenvalue of $J^0(x^*,q^*)$ to become positive under perturbations induced by diffusion. This destabilization idea is the essence of the Turing mechanism for diffusion-induced instability and pattern formation. The result of this instability could be the emergence of a regular stable patterned distribution of the state and the costate variables across the spatial domain. The Theorem below gives one set of sufficient conditions for diffusion-induced instability of optimal control.

Theorem 1 *Assume that in the problem (1)-(2), with $D = 0$, the optimal flat steady state (x^*, q^*) associated with the Jacobian matrix $J^0(x^*,q^*)$ has*

[8] See for example Murray (2003) or Okubo and Levin (2001).

[9] Kurz (1968) points out that for $\rho = 0$ it is possible to have all roots purely imaginary.

the local saddle point property. Then if

$$2H_{xq}^{0}\left(x^{*},q^{*}\right)>\rho \tag{16}$$

$$\frac{\rho^{2}}{4}>-H_{xx}^{0}\left(x^{*},q^{*}\right)H_{qq}^{0}\left(x^{*},q^{*}\right) \tag{17}$$

for $D>0$, the negative eigenvalue of $J^{0}\left(x^{*},q^{*}\right)$ becomes positive. The Jacobian matrix has both eigenvalues with positive real parts and diffusion destabilizes the stable manifold containing the optimal flat steady state.

For proof see Appendix.

As shown in the Appendix to the above Theorem, the eigenvalues associated with the spatial optimal control problem are given by

$$\lambda_{1,2}\left(k^{2}\right) = \frac{1}{2}\left(\rho\pm\sqrt{\rho^{2}-4h\left(k^{2}\right)}\right) \tag{18}$$

$$h\left(k^{2}\right) = -D^{2}k^{4}+D\left[2H_{xq}^{0}\left(x^{*},q^{*}\right)-\rho\right]k^{2}+\det J^{0}\left(x^{*},q^{*}\right) \tag{19}$$

where for a spatial domain which is a circle of length $[0,L]$, $k=2n\pi/L$, $n=\pm1,\ \pm2,...$, $1/k=L/2n\pi$ is a measure of the wave-like pattern, k is called the *wavenumber* and $1/k$ is proportional to the wavelength $\omega;\omega=2\pi/k=L/n$ here. If the conditions of the Theorem are satisfied, then $\lambda_2>0$ and the conditional stability, in the saddle point sense, is lost due to diffusion. It follows from (18) that "destabilization" of the negative eigenvalue by diffusion is equivalent to $h\left(k^{2}\right)>0$. If $h\left(k_{\max}^{2}\right)>0$, then there exist two positive roots $k_1^2<k_2^2$ such that $h\left(k^{2}\right)>0$ and $\lambda_2\left(k^{2}\right)>0$ for $k^2\in\left(k_1^2,k_2^2\right)$. As shown in the Appendix, $h\left(k_{\max}^{2}\right)=\rho^2/4+H_{xx}^{0}H_{qq}^{0}$, with $k_{\max}^{2}=\left(2H_{xq}^{0}-\rho\right)/2D$ and

$$k_{1,2}^{2}=\frac{\left(2H_{xq}^{0}-\rho\right)\pm\sqrt{\left(\rho^{2}+4\left(H_{xx}^{0}H_{qq}^{0}\right)\right)}}{2D} \tag{20}$$

The interval (k_1,k_2) determines the range of the unstable modes associated with the spatially heterogeneous solution. The function $h\left(k^{2}\right)$ is called the *dispersion relationship* associated with the optimal control problem.[10] Diffusion driven instability in the optimally controlled system emerges if the maximum of the dispersion relationship is in the positive quadrant. In honor of Turing, we call the set of parameters for which Theorem 1 is satisfied the *Turing space.*

It can easily be seen from (19) that for $D=0$, $h\left(k^{2}\right)=\det J^{0}\left(x^{*},\lambda^{*}\right)<0$

[10] For a detailed analysis of the dispersion relationship in problems without optimization, see Murray (2003, Vol II, Chapter 2).

by the saddle point property of the flat steady state, and no diffusion-induced instability is possible. However, for $D > 0$ and provided that the conditions of Theorem 1 are satisfied, we have $h'(0) = D\left(2H^0_{xq} - \rho\right) > 0$, the concave dispersion relationship is increasing at $h(0) = \det J^0 < 0$, and two positive roots emerge. It can be seen from (20) that the smaller D is, the larger these roots are. A dispersion relationship is shown in Figure 3 for the application analyzed in Section 5.1. The horizontal line corresponds to the flat case $D = 0$, but for $D > 0$, $h\left(k^2_{\max}\right) > 0$, since the conditions of Theorem 1 are satisfied. It is also clear that for $\rho = 0$, the Turing space is empty and diffusion-driven instability does not emerge. However, for higher discount rates and for appropriate values of parameters in the Jacobian matrix (15), the Turing space need not be empty, and we show that it is not empty for the economic examples discussed later on.

To get a clearer picture of the state-costate paths when the flat steady state is linearly destabilized by diffusion, we start again by analyzing the solution of the linearized MHDS for the flat case.

We know that for the optimally controlled system with $D = 0$ the optimal first approximation (linear) solution is derived from the linearized MHDS in the neighborhood of the flat steady state as:

$$\begin{pmatrix} \tilde{x}(t,z) \\ \tilde{q}(t,z) \end{pmatrix} = C_2 \mathbf{v}_2 e^{\lambda_2 t}\text{, for all } z \tag{21}$$

$$\tilde{x}(t,z) = x(t,z) - x^*,\ \ \tilde{q}(t,z) = q(t,z) - q^* \tag{22}$$

where C_2 is a constant determined by initial conditions on x and transversality conditions, and $\mathbf{v}_2$ is the eigenvector corresponding to λ_2. In particular for the linearized system the transversality condition at infinity,

$$\lim_{T\to\infty} e^{-\rho T} \int_{z_0}^{z_1} q(T,z)\, x(T,z)\, dz = 0 \text{ for all } z$$

forces the constant C_1 associated with positive root λ_1 to be zero.

The corresponding optimal first approximation to a spatially heterogeneous solution, for a linear approximation in the neighborhood of the FOSS, with the spatial domain being a circle $[0, L]$, can be written as the sum of unstable modes. As time t increases, the dominant contribution will come from those modes where $\lambda_2\left(k^2\right) > 0$, since modes with $\lambda_2\left(k^2\right) < 0$ will fade away in influence as t increases. The relevant expression, which satisfies the circle boundary conditions, is given by

$$\begin{pmatrix} \tilde{x}(t,z) \\ \tilde{q}(t,z) \end{pmatrix} \sim \sum_{n_1}^{n_2} \mathbf{B}_n \exp\left[\lambda_2\left(k^2\right)t\right]\cos\frac{2n\pi z}{L}, k^2 = \left(\frac{2n\pi}{L}\right)^2 \tag{23}$$

where the x-component of the vector $\mathbf{B}_n$, call it a_n, is determined by initial conditions on x at date $t = 0$[11] and the q-component of the vector $\mathbf{B}_n$, call it $P_n a_n$, is given by the requirement that the vector $\mathbf{B}_n := (a_n, P_n a_n)$ lie on the one-dimensional eigenspace spanned by the eigenvector with the smallest real part of the two eigenvalues corresponding to mode n, and, where $\lambda_2\left(k^2\right) > 0$ for $k^2 \in \left(k_1^2, k_2^2\right)$, n_1 is the smallest integer greater or equal to $Lk_1/2\pi$ and n_2 is the largest integer less than or equal to $Lk_2/2\pi$, and the wavenumbers k_1 and k_2 are such that $h\left(k^2\right) > 0$ for $k^2 \in \left(k_1^2, k_2^2\right)$. Since $\lambda_2\left(k^2\right) > 0$ for $k^2 \in \left(k_1^2, k_2^2\right)$, only these modes grow with time; all the remaining modes for which $\lambda_2\left(k^2\right) < 0$ tend to zero exponentially. It is important to note the significance of the size L of the spatial domain, in the emergence of diffusion-induced instability. For example the smallest wavenumber corresponds to $n = 1$. This implies that $n_1 = n_2 = 1$ and the length L should satisfy $2\pi/k_2 \leq L \leq 2\pi/k_1$. In general the size of the spatial domain should be such that allowable wavenumbers exist, and pattern formation emerges. This is made clearer in the application part, Section 5.1. Assume that the spatial domain is such that there is only one unstable wavenumber, or $n = 1$. Then the only unstable mode is $\cos(2\pi z/L)$, and the growing instability is determined by[12]

$$\begin{pmatrix} \tilde{x}(t,z) \\ \tilde{q}(t,z) \end{pmatrix} \sim \mathbf{B}_1 \exp\left[\lambda_2\left(\frac{4\pi^2}{L^2}\right)t\right]\cos\left(\frac{2\pi z}{L}\right) \tag{24}$$

where the vector of constants $\mathbf{B}_1$ is determined by initial conditions on x and by the appropriate eigenspace as above. Since the instability occurs on the stable manifold of the linearized MHDS, it would be natural to choose initial conditions on this manifold and close to the flat steady state. Take a small $\mathbf{B}_1 = (\epsilon_x, \epsilon_q)$, then using the definition of $(\tilde{x}(t,z), \tilde{q}(t,z))$ from (22), we have that the optimal spatially heterogeneous first approximation solution evolves

[11] Since our initial conditions on x are of the form $x(0,z) = x_0(z)$, we restrict ourselves to the set of initial condition functions $x_0(z)$ that possess a Fourier series of the form $\sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi z}{L}\right)$. For a treatment of regularity conditions for existence of a Fourier series, and which initial condition functions have a Fourier series with respect to a particular basis set, and whether the basis set is complete, see Priestley (1981).

[12] Initial and terminal conditions on the circle are satisfied at $z = 0$ and $z = L$.

approximately as:

$$x(t,z) \sim \epsilon_x \exp\left[\lambda_2\left(\frac{4\pi^2}{L^2}\right)t\right]\cos\frac{2\pi z}{L} + x^*, \ \frac{4\pi^2}{L^2} = k^2 \quad (25)$$

$$q(t,z) \sim \epsilon_p \exp\left[\lambda_2\left(\frac{4\pi^2}{L^2}\right)t\right]\cos\frac{2\pi z}{L} + q^* \quad (26)$$

Since $\lambda_2\left(\frac{4\pi^2}{L^2}\right) > 0$, the solution which is characterized by (25)-(26) does not decay exponentially with the passage of time, in order to converge to the flat steady state as in the no diffusion ($D = 0$) case, but it moves away from the flat state in a wave-like spatial pattern. This of course is spatial heterogeneity since the state variable and its shadow value - the costate variable - will have different values in different spatial points at any given point in time. In this case the path for the optimal control $u^*(t,z)$ in the neighborhood of the flat steady state will be determined by (7). The first approximation solution (25) is depicted in Figure 4 for the application of Section 5.1.

In this case, with a non-empty Turing space, spatially heterogenous solutions for the first approximation similar to (25)-(26) grow exponentially. This however cannot be valid for all t, since then exponential growth would imply that $(x,q) \to \infty$ at $t \to \infty$. However, the kinetics of the Hamiltonian system (8)-(9) and the transversality condition at infinity (10) should bound the solution. This suggests that the growing solution of the MHDS might settle to a spatial pattern as $t \to \infty$.[13] This implies that in the given spatial domain, the state, the costate, and the control variables will be different from the flat optimal steady-state levels. In this case an ultimate spatially Heterogeneous Optimal Steady-State solution (HOSS) for the optimally controlled system will emerge. This HOSS will satisfy

$$0 = \rho q(z) - H_x^0(x(z), q(z)) - Dq_{zz}(z) \quad (27)$$

$$0 = H_q^0(x(z), q(z)) + Dx_{zz}(z) \quad (28)$$

with the appropriate spatial boundary conditions. Setting $v = \frac{\partial x}{\partial z}$, $u = \frac{\partial q}{\partial z}$, we obtain the first-order system

$$\frac{\partial v}{\partial z} = -\frac{1}{D}H_q^0(x,q), \ \frac{\partial x}{\partial z} = v \quad (29)$$

$$\frac{\partial u}{\partial z} = \frac{1}{D}\left(\rho q - H_x^0(x,q)\right), \ \frac{\partial q}{\partial z} = u \quad (30)$$

The system can be solved with the appropriate spatial boundary conditions in

[13] See Murray (2003, Vol II, Chapter 2.4, pp. 93-94) for similar arguments.

order to determine the HOSS. More precise solutions, showing that a pattern will actually emerge at the steady state, are presented in the application part. However a further clarification might be possible here with the help of the *heterogeneity function* (Berding, 1987; Murray, 2003), which is defined as:

$$\mathcal{H} = \int_Z \left[(x_z(z))^2 + (q_z(z))^2 \right] dz \geq 0$$

Integrating by parts, using circle boundary and transversality conditions, and substituting from (27)-(28), we obtain[14]

$$\begin{aligned} \mathcal{H} &= [xx_z + qq_z]_0^L - \int_0^L (xx_{zz} + qq_{zz})\, dz \\ &= \frac{1}{D} \int_0^L \left\{ \left[x(z)\, H_q^0(x(z), q(z)) \right] - \left[q(z) \left(\rho q(z) - H_x^0(x(z), q(z)) \right) \right] \right\} dz \end{aligned}$$

It is clear that at a FOSS, $\mathcal{H} = 0$, since $H_q^0(x, q) = 0, \rho q - H_x^0(x, q) = 0$. The HOSS will exist if $\mathcal{H} > 0$.[15]

3.1 Diffusion as a Stabilizing Force

The main focus of this paper is the analysis of diffusion-induced instability as a means of spatial patterning. Diffusion, however, may act a stabilizing force for an unstable system. To show this we examine now the case where diffusion can act as a stabilizing force when the FOSS is unstable, which implies that there exist positive eigenvalues $\lambda_{1,2} > 0$ for $k^2 = 0$ in (18). Positive eigenvalues $\lambda_{1,2}$ means that, since $\mathrm{tr} J^0 = \rho > 0$, $\det J^0 > 0$. Diffusion can stabilize the system in the sense of producing a negative eigenvalue. For the smallest eigenvalue to turn negative, or $\lambda_2 < 0$, it is sufficient, from (18), that $h(k^2) < 0$. For the quadratic (19), we have that $h(0) = \det J^0 > 0$, which is the instability condition for the FOSS, and furthermore $h(k^2)$ is concave, and therefore has a maximum. Therefore there exists a root $k_2^2 > 0$,

[14] $[xx_z + qq_z]_0^L = 0$ with circle and smooth pasting boundary conditions.

[15] Murray (2003, Vol. II, Chapter 2.9) uses $\mathcal{H}$ as a Lyapunov function to show that large diffusion prevents spatial patterning in reaction diffusion mechanisms. This entails showing that for large diffusion, $d\mathcal{H}/dt$ could be negative, which implies that $\mathcal{H} \to 0$ as $t \to \infty$, which in turn implies spatial homogeneity and points to the potentially stabilizing forces of strong diffusion. We conjecture that a similar result might hold in some of the settings we examine in our applications. We suspect that such a result can be formulated and proved in the descriptive setting presented in Section 5.2.2, where there is a fixed harvesting rule in a "Solow" type model of harvesting a diffusing resource, but this is outside the scope of the present paper.

such that for $k^2 > k_2^2$, we have $(h(k^2), \lambda_2) < 0$. The solutions for $x(t,z)$ and $q(t,z)$ will be determined by the sum of exponentials of λ_1 and λ_2. Since we want to stabilize the system, we set the constant associated with the positive root λ_1 equal to zero. Then the solution will depend on the sum of unstable and stable modes associated with λ_2.

Following the previous procedure, the solutions for x and q will be of the form:

$$\begin{pmatrix} \tilde{x}(t,z) \\ \tilde{q}(t,z) \end{pmatrix} \tilde{} \sum_{0}^{n_2} \mathbf{C}_{\hat{n}} \exp\left[\lambda_2\left(\frac{4n^2\pi^2}{L^2}\right)t\right]\cos\frac{2n\pi z}{L} + \sum_{n_2}^{N} \mathbf{C}_n \exp\left[\lambda_2\left(\frac{4n^2\pi^2}{L^2}\right)t\right]\cos\frac{2n\pi z}{L}, \tag{31}$$

where n_2 is the smallest integer greater or equal to $Lk_2/2\pi$ and $N > n_2$. Since $\lambda_2\left(\frac{4n^2\pi^2}{L^2}\right) < 0$ for $n > n_2$, all the modes of the second term of (31) decay exponentially. So to converge to the steady state, we need to set $\mathbf{C}_{\hat{n}} = 0$, and have the coordinates of the vector $\mathbf{C}_n$ lie on the one-dimensional eigenspace spanned by the the eigenvector with the negative real part of the two eigenvalues corresponding to mode n. Then the spatial patterns corresponding to the second term of (31) will die out with the passage of time and the system will converge to the spatially homogeneous steady state (x^*, q^*).

4 Diffusion Driven Instability and Economic Interpretations

The results of the previous Section suggest that diffusion can destabilize a FOSS and eventually drive the MHDS to a new patterned steady state. In the general model used above we have seen that diffusion driven instability is indicated by conditions on the Hamiltonian function of the problem and its derivatives. In this Section we will try to present some economic intuition behind these conditions.

Theorem 1 presents two sufficient conditions for diffusion driven instability. Satisfaction of the second condition (17) implies the violation of the *curvature* condition derived by Brock and Scheinkman (1976) for the global asymptotic stability (GAS) of infinite horizon optimal control in the time domain only. This condition is related to the curvature matrix Q :

$$Q(x^*(t), q^*(t)) = \begin{pmatrix} -H_{xx}^0(x^*(t), q^*(t)) & \rho/2 \\ \rho/2 & H_{qq}^0(x^*(t), q^*(t)) \end{pmatrix}$$

where $(x^*(t), q^*(t))$ are solutions of the MHDS. In the scalar case if $H^0(x,q)$ is concave in x and convex in q, then $\left(-H^0_{xx}, H^0_{qq}\right) > 0$. The curvature condition implies that Q is positive definite, or

$$\frac{\rho^2}{4} < -H^0_{xx}(x^*, q^*) H^0_{qq}(x^*, q^*) \tag{32}$$

When the curvature condition is satisfied, it implies that for bounded solutions $(x^*(t), q^*(t))$ of the MHDS with $D = 0$, $(x^*(t), q^*(t)) \to (x^*, q^*)$ as $t \to \infty$, where (x^*, q^*) is the FOSS. Thus the FOSS is GAS. If we assume that $H^0(x,q)$ is α*-concave* $-$ β*-convex*, then the FOSS is GAS (Rockafellar, 1976) provided that $\rho^2 < 4\alpha\beta$. It is clear that (17) also violates the condition α*-concavity* $-$ β*-convexity*.

We can obtain a number of insights from purely formal manipulations and using the intuition behind the role of the matrix Q in analogy to Brock and Scheinkman's results in n-dimensional control systems. For example, if the Brock and Scheinkman (1976) curvature condition is satisfied, that is,

$$-(\dot{x}, \dot{q}) Q \begin{pmatrix} \dot{x} \\ \dot{q} \end{pmatrix} \leq 0 \tag{33}$$

then $\rho^2/4 < -H^0_{xx}H^0_{qq}$, or equivalently the matrix Q is positive definite. In this case the local diffusive instability condition (17) is not satisfied. We might usefully conjecture that the flat optimal steady state is GAS by an adaptation of Brock and Scheinkman's n-dimensional arguments to our infinite dimensional setting here. That is, for a bounded solution, we might conjecture that $(x^*(t,z), q^*(t,z)) \to (x^*, q^*)$ as $t \to \infty$. Furthermore, as has been shown by Feinstein and Oren (1983) in the finite dimensional case, if Q is positive definite, then the FOSS has the saddle point property. So, in terms of local analysis, satisfaction of (33) suggests that the FOSS cannot be destabilized locally by diffusion-induced perturbations. It is beyond the scope of this paper to extend the Brock-Scheinkman (1976), Cass-Shell (1976), Rockafellar (1976) type of results to the infinite horizon problems being treated here. We simply content ourselves with a purely local analysis of the first-order conditions that is parallel to Murray's (2003) treatment.

Hence, in a space time framework we conjecture that spatial heterogeneity might emerge with the violation of (33) along with the satisfaction of (16). We examine the economics behind this conjecture using the following approach.

1. Apply the envelope theorem to the maximized Hamiltonian (14) to

obtain

$$H_q^0 = g\left(x, u\left(x, q\right)\right), \; H_{qq}^0 = -g_u \frac{\partial u\left(x, q\right)}{\partial q} \tag{34}$$

It is a standard result of optimal control theory (Arrow and Kurz, 1970) that the optimal control $u^* = u(x, q)$ defined through (7) describes short-run equilibrium and can be interpreted most of the times as the short-run demand for the control u as a function of the state variable x and its shadow value, q (the costate variable), which are both treated as fixed in the short run. Thus $\partial u / \partial q$ can be interpreted as the slope of the short-run demand function along the optimal solution.

2. Take the total derivative of (8) at the FOSS to obtain

$$\rho dq + q^* d\rho - H_{xx}^0 \left(x^*, q^*\right) dx - H_{xq}^0 \left(x^*, q^*\right) dq = 0 \text{ or} \tag{35}$$

$$\frac{dx}{d\rho} = \frac{q^*}{H_{xx}^0} \tag{36}$$

In (36) $dx/d\rho$ is negative. This is the "capital deepening response" discussed by Burmeister and Turnovksy (1972). To put it another way, a FOSS is "regular" in their sense. In Burmeister and Turnovsky's "regular" case, the shadow value of capital stock falls as the interest rate increases. Here the discount rate, ρ, plays the role of the interest rate and the state variable x plays the role of capital stock. Condition (36) implies that as the discount rate on the future increases, the steady state x falls. Similarly we may interpret $dx/d\rho$ as the slope of the steady-state demand curve for the state variable (stock) at the FOSS.

Combining (34) and (36) we obtain

$$-H_{xx}^0 H_{qq}^0 = \frac{\partial u / \partial q}{\partial x / \partial \rho} \left(g_u q^*\right) \;\; or \;\; -H_{xx}^0 H_{qq}^0 = \frac{\varepsilon_{uq}}{\varepsilon_{x\rho}} \frac{\rho u^*}{x^*} \tag{37}$$

where ε_{uq} and $\varepsilon_{x\rho}$ are short-and long-run elasticities of demand with respect to the state variable shadow value and the discount rate respectively, weighted by the interest charged on the control variable per unit of the state variable. If the numerical values of the elasticities are such that the elasticities condition

$$\frac{\rho^2}{4} > \frac{\varepsilon_{uq}}{\varepsilon_{x\rho}} \frac{\rho u^*}{x^*} \tag{38}$$

is satisfied, then the FOSS is destabilized through diffusion-induced instability and a pattern emerges.

Another important factor in destabilizing the FOSS is the size of the cross partial H^0_{xq}. From (18) and (19) it is clear that since destabilization, requires $h(k^2) > 0$, H^0_{xq} should be sufficiently high. From (35) it follows that $dq/d\rho = q^*/H^0_{xq}$. Thus high H^0_{xq} means that the sensitivity of the costate variable at the FOSS to the discount rate is small. Here is another way to view the size of the cross partial H_{qx}. Brock and Malliaris (1989, Chapter 5) point out how concepts of Hamiltonian curvature (*e.g.* the Q-condition) do not capture the underlying dynamics when control is *not* applied (Brock and Malliaris (1989, page 147, eq. (6.7)), whereas the quantity H_{qx} is closely related to the underlying dynamics of the system when *no* control is applied. Hence, intuitively, one expects that the optimally controlled system is *more* likely to be stable if the underlying dynamics without control is stable. This effect is not captured by Hamiltonian curvature conditions like the Q-condition, but it is captured (at least in part) by the term H_{qx}. Given this background we can make two observations about conditions (16) and (17) of Theorem 1. First, condition (16) is close to saying that the uncontrolled system must not be "too" stable or must even be "unstable" relative to the size of ρ. Putting it another way, it says that the system must be more "unstable" the larger is ρ. Second, equation (17) essentially asserts that ρ must be larger than Hamiltonian curvature. The conclusion is that one should look out for the possibility of diffusion-induced instability in systems that are slightly "unstable" when no control is applied and where Hamiltonian curvature is weak relative to ρ. We explain in more detail below.

Since all of the parameters that appear in the key dispersion relation, $h(k^2)$, are summarized in the matrix J^0 as well as by the diffusion parameter D and the size of the space L, we provide more discussion of "comparative statics" by using subsets of these parameters, but stressing the role of D and of L. The first thing to discuss is the forces that determine whether $h(k^2)$ is positive for some k^2. That is whether there is an interval $I = (a, b)$ such that $h(k^2) > 0$ for $k^2 \in I$. The second thing to discuss is the set of values of L that allows existence of a positive integer n such that $[(2n\pi/L)]^2$ is in I. And, finally, the third thing to discuss is bifurcational loss of local stability of the FOSS and where the HOSS is likely to appear. Following Fujita, Krugman, and Venables (1999), hereafter "FKV," we shall call a HOSS an "agglomeration". However, note that our agglomeration (if it exists) is an optimal agglomeration. FKV do not treat optimization as we do here. Hence there is no role for ρ, for example, to play in their discussion. In our treatment a zero value of ρ (or a small enough value of ρ) eliminates any agglomeration. As far as we know, discussion of this role of ρ (as well as Theorem 1) in allowing (or preventing) agglomeration is new in our paper.

By graphing $h(k^2)$ against k^2, we see the following: (i) $h(0) = \det J^0$,

(ii) $h'(0) = 2H^0_{xq} - \rho$, and (iii) the maximum value of $h(k^2_{\max})$ is equal to $\rho^2/4 + H^0_{xx}H^0_{qq}$ which is the negative of the determinant of the Brock and Scheinkman curvature matrix. Thus we see that there is an interval I where $h(k^2)$ is positive, provided that $\det J^0 > 0$, or $2H^0_{xq} - \rho > 0$ and the anti Brock and Scheinkman condition holds. Of course we also need $D > 0$. If $D = 0$, $h(k^2)$ is the constant, $\det J^0$, for all values of k^2. As D goes to zero from a positive value, the function $h(k^2, D)$ becomes horizontal at the value $\det J^0$. But notice that for each positive D, the value of $\max h(k^2, D)$ (which is the negative of the determinant of the Brock and Scheinkman curvature matrix) is the same independently of $D > 0$. As D decreases towards zero, the maximum $k^2_{\max}$ and the interval I shifts off towards plus infinity. But mere existence of an interval I such that $h(k^2)$ is positive for k^2 in I is not enough for existence of diffusion-induced local instability of a FOSS. The size of the space L is crucial.

Suppose there is an interval $I(D)$ such that $h(k^2, D) > 0$ on $I(D)$ for a fixed positive value of D. The size L must be such that there is a positive integer n such that $[(2n\pi/L)]^2$ is in I. Notice that

$$\left(\frac{2\pi}{L}\right)^2 < \left(\frac{2\pi 2}{L}\right)^2 < \ldots \left(\frac{2n\pi}{L}\right)^2 < \left(\frac{2\,(n+1)\,\pi}{L}\right)^2 < \ldots, \forall\; n = 1, 2, \ldots$$

Thus we see that as L decreases, the entire set of values above shifts to the right. Therefore if L is small enough, there will be no positive integer k such that $[2\pi k/L]^2$ is in I for a fixed finite interval I. In conclusion we see that a larger space L and a smaller D make diffusion induced local instability of a FOSS more likely. Now let us think of L and D as very slowly changing variables relative to the dynamics of the optimization problem. As L slowly increases and D slowly decreases, a FOSS becomes locally unstable provided that $2H^0_{xq} - \rho > 0$ and $\rho^2/4 + H^0_{xx}H^0_{qq} > 0$. That is, in our optimization system, much like FKV (1999), we have a "bifurcation" of a "flat earth" optimum to form an "optimal agglomeration," *i.e.* a HOSS in our jargon. The same type of discussion of path dependence on this very slow time scale as in FKV (1999, Chapter 17) applies here too. We believe there is a potentially very fruitful research program in extending the theory in our paper to optimized versions of FKV systems, but that task is way beyond the scope of the current paper.

Finally it should be noticed that to create a pattern in the optimal control model, there is no need to assume increasing returns. We mention this because it is common to assume zones of increasing returns in the more recent literature on spatial economics and agglomeration. Zones of increasing returns will be assumed in some of the applications below.

5 Applications

The previous Sections developed a general theoretical framework of optimal control in time space under diffusion. We explored conditions under which optimal control in time and space could lead to pattern formation and spatial heterogeneity at a steady state. In this Section we present some economic applications. These applications are very abstract and highly stylized. But we think they are useful in illustrating the potential usefulness of analysis of diffusion driven spatial stability or instability in dynamic economics.

5.1 Optimal Management and Spatial Pattern Formation in Shallow Lakes

Our first example is a stylized version of the "lake" problem (Carpenter, Ludwig, and Brock, 1999; Brock and Starrett, 2003; Mäler, Xepapadeas, and de Zeeuw, 2003), but where the dynamics of the lakes are connected by diffusion of pollutants, *e.g.*, inflow of unwanted nutrients for algae, across lakes. We will call this problem the "shallow lake" problem because damaging changes to alternative stable states have actually been documented by experiments with shallow lakes (cf. references to the limnological literature in Carpenter et al. (1999)). Here the problem is to maximize the discounted stream of services from an ecosystem, *e.g.*, a stylized lake, whose dynamics are nonlinear where there is a trade-off between "loaders" and those who enjoy the amenities of an unspoiled ecosystem. The loaders include industry, agriculture, green lawn owners, leaking septic tanks of cottage owners, etc., and the enjoyers of the amenities provided by the lake include fishermen, swimmers, sail boaters, birders, hikers, other types of viewers, etc. The mechanism of damage to the lake is nutrient inflow from loaders which causes algae to multiply which causes noxious blooms, fish kills, unpleasant smells, etc. Following Carpenter et al. (1999), we use a stylized on-dimensional state variable called "phosphorous sequestered in algae" to summarize the state of each lake. The dynamics of each lake allow the possibility of alternative stable states, one of which is called "oligotrophic" and which is characterized by low phosphorous sequestered in algae and is "good"; the other is called "eutrophic" and is characterized by low phosphorous sequestered in algae, and is "bad".

Carpenter and Brock (2003) study a system of spatially connected lakes where the collapse of one lake stimulates negative spillover effects into other nearby lakes. Their system is much more realistic, their mechanism of spillover is different from ours, and their model is much more complicated than the stylized system of spatially connected lakes that we study here. For example the only type of spillover we allow here is the spillover of unwanted nutrients from one lake into neighboring lakes. Furthermore we represent "space" as a ring of lakes which are connected like a chain of beads on that ring. Nevertheless we can obtain some understanding of how the interaction of the discount rate with the curvature of the Hamiltonian function can lead to diffusion-induced instability, *i.e.* a "domino" effect, using the methods developed in our paper, whereas Carpenter and Brock (2003) used "movies" and numerical methods to uncover the causes of potential domino effects in their system. We believe understanding is enhanced using both types of methods, that is: (i) Numerical analysis of more realistic, but analytically intractable spatially connected dynamical systems, and (ii) Strategically chosen severe abstractions of spatially connected dynamical systems which are still analytically tractable using methods like those developed in this paper.

The spatial optimal management 'shallow lake' problem can be stated as:

$$\begin{aligned}
&\max_{a(t,z)} \int_0^\infty \int_0^L e^{-\rho t} \left[B\left(a\left(t,z\right)\right) - C\left(x\left(t,z\right)\right)\right] dzdt \\
&s.t.\ \frac{\partial x\left(z,t\right)}{\partial t} = a\left(t,z\right) - bx\left(t,z\right) + h\left(x\left(t,z\right)\right) + D\frac{\partial x^2\left(z,t\right)}{\partial z^2} \qquad (39) \\
&x\left(t,0\right) = x\left(t,L\right),\ \text{and smooth pasting}
\end{aligned}$$

where $a\left(t,z\right)$ is phosphorous loading at time t and spatial point $z \in [0,L]$ of a one-dimensional spatial domain, and $x\left(t,z\right)$ is the corresponding stock of phosphorus measured as phosphorous sequestered in algae as in Carpenter et al. (1999), taking non-negative values in a compact set $\mathcal{X}$. Here $B\left(a\left(t,z\right)\right)$ denotes concave benefits to the 'loaders' and $D\left(x\left(t,z\right)\right)$ denotes convex damages to those enjoying the lake's amenities. In (39), which describes the evolution of the phosphorous stock, b is the rate of loss per unit stock (*e.g.* sedimentation, outflow) and the function $h\left(x\left(t,z\right)\right)$ is a positive feedback representing internal loading which is assumed to be "S-shaped". The generalized current value Hamiltonian of this problem is

$$\tilde{H}\left(a,x,\lambda\right) = B\left(a\right) - C\left(x\right) + \lambda\left[a - bx + h\left(x\right) + D\frac{\partial x^2\left(z,t\right)}{\partial z^2}\right]$$

Using the necessary conditions implied by the Maximum Principle under diffusion (MPD-NC) we obtain

$$
\begin{aligned}
& B^{'}(a) = -\lambda \Rightarrow a = \bar{a}(\lambda), \quad \frac{da}{d\lambda} = -\frac{1}{B''} > 0 \\
& \frac{\partial x(z,t)}{\partial t} = H_\lambda^0(x,\lambda) = \bar{a}(\lambda) - bx + h(x) + D\frac{\partial x^2(z,t)}{\partial z^2} \qquad (40) \\
& \frac{\partial \lambda(z,t)}{\partial t} = \left[\rho + b - h^{'}(x)\right]\lambda + C^{'}(x) - D\frac{\partial \lambda^2(z,t)}{\partial z^2} \qquad (41)
\end{aligned}
$$

The FOSS (x^*, λ^*) is determined by the solution of the system

$$
\begin{aligned}
& 0 = \bar{a}(\lambda) - bx + h(x) = H_\lambda^0(x,\lambda) \\
& 0 = \left[b + \rho - h^{'}(x)\right]\lambda + C^{'}(x) = \rho\lambda - H_x^0(x,\lambda) \\
& H^0(x,\lambda) = \max_a H(a,x,\lambda)
\end{aligned}
$$

So the FOSS is characterized by:

$$
\begin{aligned}
\bar{a}(\lambda) &= bx - h(x) \\
\lambda|_{\dot{\lambda}=0} &= \frac{-C^{'}(x)}{\left[b + \rho - h^{'}(x)\right]} \\
\bar{a}\left(\frac{-C^{'}(x^*)}{\left[b + \rho - h^{'}(x^*)\right]}\right)\Bigg|_{\dot{x}=0} &= bx^* - h(x^*) \Rightarrow x^* = X(\rho, b) \\
\lambda^* &= \frac{-C^{'}(X(\rho))}{\left[b + \rho - h^{'}(X(\rho,b))\right]} = \Lambda(\rho, b)
\end{aligned}
$$

The stability of the FOSS depends upon the Jacobian evaluated at the FOSS, or

$$
\begin{aligned}
J^0 &= \begin{pmatrix} H_{\lambda x}^0 & H_{\lambda\lambda}^0 \\ -H_{xx}^0 & \rho - H_{x\lambda}^0 \end{pmatrix} = \begin{pmatrix} -b + h^{'}(x^*) & -1/B'' \\ -h''(x^*)\lambda^* + C''(x^*) & b + \rho - h^{'}(x^*) \end{pmatrix} \\
&= \begin{pmatrix} S & G \\ Q & \rho - S \end{pmatrix}
\end{aligned}
$$

We have $\mathrm{tr} J^0 = \rho > 0$ and $\det J^0 = \left(-b + h^{'}(x^*)\right)\left(b + \rho - h^{'}(x^*)\right) - \frac{\left(-h''(x^*)\lambda^* + C''(x^*)\right)}{B''}$.

The sufficient conditions for local diffusion-induced instability according to Theorem 1 are:

$$\begin{aligned}\det J^0 &< 0 && (42)\\ 2S-\rho &> 0 \; or \; 2\left(-b+h^{'}\left(x^*\right)\right)-\rho>0 && (43)\\ \frac{\rho^2}{4}-QG &> 0 \; or \; \frac{\rho^2}{4}-\frac{-\left(-h^{''}\left(x^*\right)\lambda^*+C^{''}\left(x^*\right)\right)}{B^{''}\left(a^*\right)}>0 && (44)\end{aligned}$$

It should be noted that if there is no feedback, that is, $h(x)\equiv 0$, then $\det J^0 = -b(b+\rho)+\frac{C^{''}(x^*)}{B^{''}}<0$, and the FOSS is a saddle point. Since $2S-\rho=-2b-\rho<0$, local diffusion-induced instability is not possible in this case, and the Turing space is empty.

With non zero feedback the conditions for diffusion-induced instability can also be expressed in terms of marginal benefits and marginal damages. We can gain some intuition by expressing these conditions in terms of slopes of marginal benefits and slopes of marginal damages at the FOSS, which in turn can be expressed as:[16]

$$\begin{aligned}\rho-2\frac{C^{'}\left(x^*\right)}{B^{'}\left(a^*\right)} &> 0\\ \frac{\rho^2}{4}+\frac{\left[h^{''}\left(x^*\right)B^{'}\left(a^*\right)+C^{''}\left(x^*\right)\right]}{B^{''}\left(a^*\right)} &> 0\end{aligned}$$

To clarify whether diffusion-induced spatial instability actually emerges in the optimal control of a stylized shallow lake, we examine the following specification for the problem.[17]

$$\begin{aligned}B(a) &= \ln a,\; C(x)=\frac{1}{2}cx^2,\; h(x)=\frac{x^2}{1+x^2},\; c=0.5\\ \dot{x} &= a-bx+\frac{x^2}{1+x^2}+Dx_{zz}\,,\; b=0.54,\; \rho=0.1273\end{aligned}$$

The value of $b=0.54$ corresponds, for the flat case $(D=0)$, to a reversible shallow lake with hysteresis. This is shown in Figure 1, which depicts, in the (x,a) space, the curve of the lake equilibria, defined by $\dot{x}=0$ in (39), for fixed loadings a. For loadings below the loading corresponding to the

[16] The conditions can be obtained by noting that from $0=\left[\rho+b-h^{'}\left(x^*\right)\right]\lambda^*+D^{'}\left(x^*\right)$ and $B^{'}\left(a^*\right)=-\lambda^*$, we have $D^{'}\left(x^*\right)=\left[\rho+b-h^{'}\left(x^*\right)\right]B^{'}\left(a^*\right)$ or $\frac{D^{'}(x^*)}{B^{'}(a^*)}=-S+\rho$ or $S=\rho-\frac{D^{'}(x^*)}{B^{'}(a^*)}$, $a^*=\bar{a}\left(\lambda^*\right)$.

[17] See Mäler et al. (2003) for the functional forms.

local maximum of this curve, the lake remains in a locally stable oligotrophic state. A small increase in loading above the local maximum will cause the lake to flip to a locally stable eutrophic state. To bring the lake back to the oligotrophic state, the loading has to be reduced sufficiently to a level below the local minimum of the lake equilibria curve. This is the hysteresis effect.

Figure 1

The FOSSs for this problem are characterized by

$$\begin{aligned} \frac{1}{a} &= -\lambda \\ \phi(x)|_{\dot{x}=0} &: \quad \lambda = \frac{1}{-bx + h(x)} \\ \psi(x)|_{\dot{\lambda}=0} &: \quad \lambda = \frac{-cx}{b + \rho - h'(x)} \end{aligned} \tag{45}$$

There are three FOSSs as shown in Figure 2 below. The dashed line corresponds to the $\phi(x)|_{\dot{x}=0}$ curve, while the solid corresponds to the $\psi(x)|_{\dot{\lambda}=0}$ curve. Comparing to Figure 1, the $\phi(x)|_{\dot{x}=0}$ curve corresponds to the lake equilibria curve when optimal loading is applied at the state-costate space (x, λ). It can be seen that when the oligotrophic steady state is very close to the flip point, a small perturbation can move the system to the eutrophic steady state.

Figure 2

The Jacobian at the FOSSs corresponding to (45) is

$$J_s^0 = \begin{pmatrix} -b + h'(x^*) & 1/(\lambda^*)^2 \\ -h''(x^*)\lambda^* + c & b + \rho - h'(x^*) \end{pmatrix}$$

and FOSSs are characterized in Table 1.

Table 1: FOSS for the Shallow Lake

i	$(x^*, \lambda^*)_i$	Eigenvalues	$\det J^0$	Stability
1	$(0.571, -16.019)$	$0.13542, -0.00812$	-0.00109	Saddle Point
2	$(0.583, -16.350)$	$0.11833, \ 0.00897$	0.00106	Unstable
3	$(2.445, -2.157)$	$0.62449, -0.49719$	-0.31049	Saddle Point

There are two locally stable basins of attraction:. a eutrophic (high phosphorus stock) and an oligotrophic (low phosphorus stock), with the expected high and low (in absolute terms) shadow phosphorus cost reflected by λ.

Spatial perturbation due to movement of the phosphorous stock in the lake might destabilize, through diffusion-induced instability, the first or the third FOSS. The conditions for a non empty Turing space and the corresponding values for the chosen specification are presented in Table 2.

Table 2: Emergence of Spatial Heterogeneity in the Shallow lake

Diffusion-induced instability conditions	Value at $(x^*, \lambda^*)_i$	
(43) and (44)	$i = 1$	$i = 3$
$2H^0_{x\lambda} - \rho = 2\left[-b + h'(x^*_i)\right] - \rho > 0$	0.091633	−1.00648
$\frac{\rho^2}{4} + H^0_{xx}H^0_{\lambda\lambda} = \frac{\rho^2}{4} + \left[\frac{h''(x^*_i)}{\lambda^*} - \frac{c}{(\lambda^*)^2}\right] > 0$	0.00100	−0.057241

As shown by our results, the phosphorus movements can produce spatial patterns for the optimally managed lake only if the lake is in the oligotrophic state. The elasticity condition (38) implies that pattern formation requires that the ratio of the elasticity of the demand for phosphorus loading with respect to the phosphorus stock's shadow cost, to the elasticity of the phosphorus stock with respect to the discount rate, multiplied by the interest charge on loadings per unit of phosphorus stock be sufficiently small relative to the discount rate. If the lake has flipped to a eutrophic state, then even with movements of phosphorus the optimal strategy suggests a spatially homogenous outcome in the sense that the FOSS is locally asymptotically stable. We turn now to the case where the FOSS is oligotrophic.

The dispersion relationship (19) corresponding to the oligotrophic FOSS is $h(k^2) = -D^2k^4 + 0.091633Dk^2 - 0.00109891$; it is shown in Figure 3 for values of $D = \{0, 0.025, 0.05, 1\}$. The dispersion relationship corresponding to $D = 0$ is the horizontal line at $h = -0.00109891$, and the concave curve closest to the vertical axis is the one corresponding to $D = 1$.

Figure 3

We choose $D = 0.5$ with $(k_1^2, k_2^2) = (0.168463, 0.393556)$. To have one unstable mode in the linear approximation solution (24), for the growing instability in the neighborhood of the oligotrophic FOSS, the length of the spatial domain should satisfy, as shown in Section 3, $2\pi/k_2 \leq L \leq 2\pi/k_1$, which implies that $15.9652 \leq L \leq 37.2971$. Choosing $L = 8\pi$, the positive eigenvalue is $\lambda_2 = 0.00652502$, the growing instability is given by $x(t, z) \sim \epsilon_x \exp[0.00652502t] \cos \frac{2\pi z}{L} + 0.571$ for small ϵ_x, and is shown in Figure 4.

Figure 4

Once the oligotrophic FOSS is destabilized through diffusion, and its linear approximation evolves similar to the shape shown in Figure 2, the question is whether the nonlinear ecosystem will end up, bounded by the intertemporal transversality condition, in a new steady state characterized by spatial heterogeneity and a persistent spatial pattern. The heterogeneous optimal steady state (HOSS) is characterized by

$$
\begin{aligned}
D\frac{\partial x^2(z,t)}{\partial z^2} &= -\bar{a}(\lambda) + bx - h(x) \\
D\frac{\partial \lambda^2(z,t)}{\partial z^2} &= \left[b + \rho - h'(x)\right]\lambda + D'(x)
\end{aligned}
$$

or

$$
\begin{aligned}
\frac{dv(z)}{dz} &= \frac{1}{D}\left(\frac{1}{\lambda(z)} + bx(z) - h(x(z))\right), \quad \frac{dx(z)}{dz} = v(z) \\
\frac{dw(z)}{dz} &= \frac{1}{D}\left[\left(b + \rho - h'(x(z))\right)\lambda(z) + cx(z)\right], \quad \frac{d\lambda(z)}{dz} = w(z)
\end{aligned}
$$

with boundary conditions derived by the circle and transversality conditions $x(0) = x(L) = 0.571396$, $\lambda(0) = \lambda(L) = -16.0194$, $v(0) = v(L)$, $w(0) = w(L)$ and $L = 8\pi$. The system was solved numerically by a multiple shooting approach using Mathematica. The spatial paths for phosphorus stock and its shadow cost have very flat U shapes.[18]

5.2 Spatial Pattern Formation in Natural Resource Management

Let $x(t,z)$ denote the concentration of the biomass of a renewable resource (*e.g.* fish) at spatial point $z \in [0,L]$ of a one-dimensional spatial domain at time t, with x taking non-negative values in a compact set $\mathcal{X}$. As is well-known (*e.g.* Clark 1990), natural resources can be interpreted as capital assets since the resource's value is equal to the present value of the flow of net future expected benefits. We will exploit this capital-theoretic aspect of resource management to explore diffusion-induced instability and pattern formation in renewable management by using two approaches. The first uses a standard Schaefer structure that includes stock effects. The second mimics growth theory and explores pattern formation in resource models which have the structure of the traditional Ramsey and Solow models of growth theory, without stock effects in the objective function. Boundary conditions could

[18] A more profound pattern emerges in the next application.

be either circle, or zero flux, or hostile boundary type. In these examples we concentrate on circle boundary conditions, since we want to study endogenous pattern formation. In both cases we identify conditions for diffusion-induced instability and pattern formation.

5.2.1 Optimal resource management with stock effects

Let the evolution of the biomass in space-time depend on resource growth, according to a standard concave growth function $f(x(t,z))$, dispersion in space with a constant diffusion coefficient D, and harvesting at a rate $h(t,z)$. We assume that the movement of the biomass is from high concentration towards low concentration. Stock effects are introduced by specifying harvesting as $h(t,z) = qE(t,z)\,x(t,z)$, where q is the catchability coefficient and $E(t,z)$ is harvesting effort at time t and location z.[19] Therefore,

$$\frac{\partial x(t,z)}{\partial t} = f(x(t,z)) - h(t,z) + D\frac{\partial^2 x(t,z)}{\partial z^2}$$

We assume that net harvesting benefits at each point in space-time can be represented by an increasing and concave net benefit function $Q(qEx)$. The optimal harvesting problem in space-time is then defined as:

$$\max_{E(t,z)} \int_0^{\infty}\int_{\mathcal{Z}} e^{-\rho t} Q\left(qE(t,z)\,x(t,z)\right) dz dt \tag{46}$$

$$\text{s.t. } \frac{\partial x(t,z)}{\partial t} = f(x(t,z)) - qx(t,z)\,E(t,z) + D\frac{\partial^2 x(t,z)}{\partial z^2} \tag{47}$$

$$x(t,0) = x(t,L) \text{ and smooth pasting} \tag{48}$$

The generalized current value Hamiltonian for this problem is defined as:

$$\tilde{H} = Q(qEx) + \mu\left[f(x) - qEx + D\frac{\partial^2 x(t,z)}{\partial z^2}\right]$$

[19] See for example Neubert (2003) for a fishery model with a similar structure. For renewable resource management with metapopulation models in a patchy environment, see for example Sanchirico and Wilen (2005).

Following our results in Section 2, the optimality conditions are:

$$\begin{aligned} qQ^{'}x &= q\mu x \Rightarrow Q^{'} = \mu \Rightarrow E = E\left(x,\mu\right) & (49)\\ & Q^{''}xdE + Q^{''}Edx = d\mu \\ \frac{\partial E}{\partial x} &= -\frac{E}{x} < 0 \ , \ \frac{\partial E}{\partial \mu} = \frac{1}{Q^{''}x} < 0 \\ \dot{x} &= f\left(x\right) - qE\left(\mu,x\right)x + Dx_{zz} & (50)\\ \dot{\mu} &= \left[\rho - f^{'}\left(x\right) + qE\left(x,\mu\right)\right]\mu - qQ^{'}E\left(x,\mu\right) - D\mu_{zz} \ or & (51)\\ \dot{\mu} &= \left[\rho - f^{'}\left(x\right)\right]\mu - D\mu_{zz} \text{ using } Q^{'} = \mu \end{aligned}$$

The FOSS is defined for $D = 0$ as:

$$0 = f\left(x^*\right) - qE\left(\mu^*, x^*\right)x^* = H_\mu^0\left(x,\mu\right) \tag{52}$$

$$0 = \rho - f^{'}\left(x^*\right) = \rho - H_x^0\left(x,\mu\right) \tag{53}$$

$$H^0\left(x,\mu\right) = \max_E\left(x,\mu,E\right) \tag{54}$$

Linearization at the FOSS results in[20]

$$\begin{aligned} J^0 &= \begin{pmatrix} H^0_{\mu x} & H^0_{\mu\mu} \\ -H^0_{xx} & \rho - H^0_{x\mu} \end{pmatrix} = \begin{pmatrix} f^{'}\left(x^*\right) & \frac{-q}{Q^{''}} \\ -f^{''}\left(x^*\right)\mu^* & \rho - f^{'}\left(x^*\right) \end{pmatrix} = \\ & \begin{pmatrix} S & G \\ Q & \rho - S \end{pmatrix} = \begin{pmatrix} \rho & \frac{-q}{Q^{''}} \\ -f^{''}\left(x^*\right)\mu^* & 0 \end{pmatrix} \end{aligned} \tag{55}$$

Since $\mathrm{tr}J^0 = \rho >$ and $\det J^0 = -(qf^{''}\left(x^*\right)\mu^*)/Q^{''} < 0$, the FOSS is a local saddle point for $\mu^* > 0$. From (55), $S = \rho$ and (16) is satisfied for $\rho > 0$. Condition (17), the violation of the 'curvature assumption', requires $(\rho^2/4) > -H^0_{xx}H^0_{\mu\mu}$, or

$$\frac{\rho^2}{4} > \left(-f^{''}\left(x^*\right)\mu^*\right)\left(\frac{-q}{Q^{''}}\right) \tag{56}$$

If (56) is also satisfied, the stable manifold of the saddle point FOSS is destabilized by biomass movement, through diffusion-induced instability and optimal control interactions and a spatial pattern for the biomass concentration emerges.

The elasticity condition for diffusion-induced instability implies in this case that if $\rho^2/4 > \frac{\varepsilon_{E\mu}}{\varepsilon_{x\mu}}q\rho E^*$, then optimal harvesting of the moving biomass

[20] Since $H_{\mu x} = f^{'}\left(x\right) - qE\left(\mu,x\right) - qx\frac{dE}{dx} = f^{'}\left(x\right) - qE\left(\mu,x\right) - qx\left(-\frac{E}{x}\right) = f^{'}\left(x\right).$

leads to spatial patterns, where $\varepsilon_{E\mu}$ is the demand elasticity for effort, $\varepsilon_{x\mu}$ is the elasticity of demand for *in situ* biomass with respect to the discount rate, and $(q\rho E^*)$ is a constant evaluated at the steady state reflecting the interest charged on effort multiplied by the catchability coefficient. Thus if the effort demand elasticity is small, the biomass demand elasticity is large, the interest charged on effort weighted by the catchability coefficient is not high, and the discount rate is sufficiently high, then biomass movements might destabilize the FOSS and create spatial patterns.

Pattern formation and the heterogeneous optimal steady state To examine whether there is a parameter set satisfying (56), or to put it differently, to examine whether the Turing space is non-empty, we consider again a numerical example. Assume a standard logistic model, $f(x) = rx\left(1 - \frac{x}{K}\right)$, where r is the intrinsic growth rate and K the carrying capacity, with $r = 0.08$ and $K = 400000$ whales.[21] Assuming a quadratic benefit function $Q(h) = A + \alpha h - (1/2)\beta h^2$, $h = qEx$, we obtain:

$$E(t,z) = \frac{\alpha - \mu(t,z)}{q\beta x(t,z)}$$

For the FOSS we obtain, using (52)-(53),

$$x^* = \frac{K(r-\rho)}{2\rho}, \quad \mu^* = \frac{r(2\alpha - K) + K\rho}{\beta r(r+\rho)}$$

Taking $\alpha = 80000$, $\beta = 10$, $q = 1$, the relationship $g(\rho) = \rho^2/4 - \left(-f''(x^*)\mu^*\right)\left(-1/Q''\right)$ is depicted in Figure 3. Diffusion-induced instability emerges when $g(\rho) > 0$. Choosing $\rho = 0.05$, so that $g(0.05) > 0$, we obtain for the FOSS $(x^*, \mu^*) = (75000, 7692.31)$.

Figure 3

The HOSS is characterized by

$$\frac{\partial^2 x}{\partial z^2} = \frac{1}{D}\left[E(\mu, x)x - f(x)\right] \tag{57}$$

$$\frac{\partial^2 \mu}{\partial z^2} = \frac{1}{D}\left[\rho - f'(x)\right]\mu \tag{58}$$

Making the substitutions $\frac{\partial x}{\partial z} = v(z)$, $\frac{\partial \mu}{\partial z} = w(z)$ and using the circle boundary conditions and the appropriate transversality conditions $x(0) = x(L) =$

[21] These parameters correspond to the Antarctic fin-whale (Clark, 1990 page 49).

75000, $\mu(0) = \mu(L) = 7692.31$, $v(0) = v(L)$, $w(0) = w(L)$, we solve the system by multiple shooting. Figure 4 shows the spatial paths for x and μ at the HOSS for $L = 2\pi$. The U curve is the spatial path for the biomass stock, while the lower curve which has a very flat inverted U shape is the spatial path for the resource's user cost.

Figure 4

5.2.2 Ramsey/Solow Capital Theoretic Models of Renewable Resource Management

The capital theoretic structure of this model implies that the evolution of the biomass in space-time is described by

$$\frac{\partial x(z,t)}{\partial t} = f(x(z,t)) - \delta x(z,t) - h(z,t) + D\frac{\partial x^2(z,t)}{\partial z^2}$$

where $f(x(z,t))$ is a concave growth function, δ reflects mortality rate, $h(z,t)$ is harvest rate and dispersion of the resource in space is determined by the diffusion coefficient D. We analyze first a Ramsey-type optimal resource management problem.

Pattern formation in Optimal Resource Management Consider the spatial optimal resource management problem

$$\max_{\{h(t,z)\}} \int_0^\infty \int_0^L e^{-\rho t} U(h(z,t))\, dzdt$$

$$\begin{aligned} s.t.\ \frac{\partial x(z,t)}{\partial t} &= f(x(z,t)) - h(z,t) - \delta k(z,t) + D\frac{\partial x^2(z,t)}{\partial z^2} \\ x(t,0) &= x(t,L),\ \text{and smooth pasting} \end{aligned}$$

where $U(\cdot)$ is a standard utility function reflecting the flow of net benefits from harvesting. This problem has the structure of an optimal growth model and corresponds to an optimal fishery management problem without resource stock externalities affecting the objective function. Resource stock externalities are likely to be negligible if the size of the fish population is very large relative to the industry (Smith, 1969). Using again the necessary conditions for the Maximum Principle under diffusion(MPD-NC), we obtain

$$\tilde{H}(x,h,p) = U(x(z,t)) + p(z,t)\left[f(x(z,t)) - h(z,t) - \delta x(z,t) + D\frac{\partial x^2(z,t)}{\partial z^2}\right]$$

$$
\begin{aligned}
& U_h(h(z,t)) = p(z,t) \Rightarrow h(z,t) = \bar{h}(p(z,t)), \ \frac{dh}{dp} = \frac{1}{U_{hh}} < 0 && (59)\\
& \frac{\partial x(z,t)}{\partial t} = f(x(z,t)) - \bar{h}(p(z,t)) - \delta x(z,t) + D\frac{\partial x^2(z,t)}{\partial z^2} \\
& \frac{\partial p(z,t)}{\partial t} = (\rho + \delta - f_x(x(z,t)))\, p(z,t) - D\frac{\partial p^2(z,t)}{\partial z^2} && (60)
\end{aligned}
$$

The (FOSS) (x^*, p^*) is determined by the solution of the system

$$
\begin{aligned}
0 &= f(x) - \bar{h}(p) - \delta x = H_p^0(x,p) \\
0 &= (\rho + \delta - f_x(x))\, p = \rho - H_x^0(x,p) \\
H^0(x,p) &= \max_h H(x,h,p)
\end{aligned}
$$

and the FOSS is given by

$$
\begin{aligned}
x^* &: \ f_x(x^*) = \rho + \delta \\
p^* &: \ \bar{h}(p^*) = f(x^*) - \delta x^*
\end{aligned}
$$

Assuming a Cobb-Douglas growth function $f(x) = x^a$,[22] $0 < a < 1$, and $U(h) = \frac{h^{1-\theta}-1}{1-\theta}$, $\theta > 1$ we obtain

$$
\begin{aligned}
\bar{h}(p) &= p^{-1/\theta} \\
x^* &= \left(\frac{\rho+\delta}{a}\right)^{\frac{1}{a-1}} \\
p^* &= \left[(x^*)^a - \delta x^*\right]^{-\theta}
\end{aligned}
$$

The stability of the FOSS is determined by the Jacobian matrix

$$
\begin{aligned}
J^0 &= \begin{pmatrix} H_{px}^0 & H_{pp}^0 \\ -H_{xx}^0 & \rho - H_{xp}^0 \end{pmatrix} = \begin{pmatrix} f_x(x^*) - \delta & -\bar{h}'(p^*) \\ -f_{xx}(x^*)\, p^* & 0 \end{pmatrix} = \\
& \begin{pmatrix} \rho & \frac{1}{\theta} p^{-\left(\frac{1}{\theta}+1\right)} \\ -(a-1)\, a\, (x^*)^{(a-2)}\, p^* & 0 \end{pmatrix} = \begin{pmatrix} S & G \\ Q & \rho - S \end{pmatrix}
\end{aligned}
$$

It is clear that: trace$J^0 = \rho > 0$, $\det J^0 = \frac{-(a-1)a(x^*)^{(a-2)}(p^*)^{-1/\theta}}{\theta} < 0$. There-

[22] Our Cobb-Douglas assumption could be a convenient parametrization under the assumption that steady states occur within the upward sloping part of the growth law. To have finite carrying capacity, we can paste onto the Cobb-Douglas function for large x an increasing and then a decreasing part, which would give a large carrying capacity. This part, however, does not enter our solution for the parameter constellation we are using for our illustrative example.

fore the FOSS is a saddle point.

To destabilize, according to Theorem 1, the negative root through diffusion-induced instability via optimal control interactions, we need

$$\begin{aligned} 2S-\rho &> 0,\ or\ \rho>0 \\ \frac{\rho^2}{4}-QG &> 0\ or \end{aligned} \tag{61}$$

$$g(\rho) = \rho^2+4\frac{(a-1)\,a\,(x^*)^{(a-2)}\,(p^*)^{-1/\theta}}{\theta}>0 \tag{62}$$

For $\delta=0.01, \theta=3, a=0.65$, relation (62) implies, as in the previous case, that diffusion-induced instability emerges when $\rho>0.037$. Choosing $\rho=0.25$, the corresponding FOSS is $(x^*,\mu^*)=(13.708, 0.0065)$.

To explore the economic intuition as before, the elasticity condition (38) implies

$$\frac{\rho^2}{4}>-H^0_{pp}H^0_{xx}=p\frac{d\bar{h}/dp}{dx/d\rho}=\frac{\varepsilon_{hp}}{\varepsilon_{x\rho}}\frac{\rho h^*}{x^*} \tag{63}$$

where ε_{xp} is the consumer demand elasticity for harvested biomass, $\varepsilon_{x\rho}$ is the elasticity of demand for *in situ* biomass with respect to the discount rate, and $(\rho h^*/k^*)$ is a constant evaluated at the steady state reflecting the interest charged on harvested biomass per unit of biomass remaining *in situ*. Actually ρh^* can be interpreted as forgone biomass returns due to the fact that the biomass did not remain *in situ*, but was removed by harvesting. Thus (63) implies that if the consumer demand elasticity is small the biomass demand elasticity is large, the interest charged on harvested biomass per unit of *in situ* biomass at the FOSS is not high, and the discount rate is sufficiently high, then biomass movements might destabilize the FOSS and create spatial patterns.

The (HOSS), approximated again by multiple shooting, implies a very flat U curve as the spatial path for the biomass and a very flat inverted U curve as the spatial path for its shadow value.

Pattern formation with a fixed harvesting rule Assume that fishery management is characterized by a fixed harvesting rule which states that a fixed proportion $1-s$, $s\in(0,1)$ of the gross additions to biomass is harvested at any point in time. In this case biomass evolution is described by

$$\frac{\partial x(z,t)}{\partial t}=sf(x(z,t))-\delta x(z,t)+D\frac{\partial x^2(z,t)}{\partial z^2}$$

where the fixed harvesting rule is common to all locations.[23] It is a routine result, with the Cobb-Douglas growth function used above, that a non zero flat steady-state (FSS) is defined for $D = 0$ as: $x^* = (\delta/s)^{1/(a-1)}$, with $a \in (0,1)$ for concavity which implies diminishing returns in biomass growth. It is also a standard result, that the FSS is asymptotically stable for positive values of biomass.[24] Let us consider the impact of the diffusion-induced perturbation on this FSS. The linearized model is

$$\begin{aligned} \frac{\partial x(z,t)}{\partial t} &= \left[sf'(x^*) - \delta \right] (x(t,z) - x^*) + D \frac{\partial x^2(z,t)}{\partial z^2} \\ x(t,0) &= x(t,L) \end{aligned} \tag{64}$$

We look for solution $(x(z,t) - x^*) \propto e^{\lambda t} \cos(2n\pi z/L)$, $n = 1, 2, \ldots$. Substituting into (64) and canceling equal terms, we obtain

$$\lambda = \left[(a-1)\delta - D(2n\pi/L)^2 \right] < 0 \tag{65}$$

Therefore $(x(z,t) - x^*) \to 0$ as $t \to \infty$, and diffusion can not produce spatial heterogeneity in the fixed harvesting rule model, with diminishing returns to biomass.

It should be noticed that, since the emergence of diffusion requires that $\lambda > 0$, this is possible only if $\alpha > 1$. Even with $\alpha = 1$, which corresponds to a linear growth function, spatial heterogeneity is not possible. So for the emergence of spatial heterogeneity in the fixed harvesting model, we need convex growth, that is increasing returns which should be sufficiently high to overcome the impact of diffusion. It follows from (65) that if D is large, then λ could be negative even with $a > 1$, in which case spatial patterns die out and the system converges to a FSS, despite increasing returns. In this case large diffusion slows down the impact of increasing returns and induces space homogeneity.

Considering that convex growth is unlikely in the typical fishery model, this result suggests that fixed harvesting rules tend to produce spatially homogeneous biomass distribution, while the optimal harvesting rule is likely, under a certain parameter constellation, to result in spatially heterogeneous biomass distribution. This heterogeneity result does not require increasing returns.

[23]The structure of this model is similar to the Solow growth model with fixed savings ratio.

[24]The linearization around the FSS results in $\dot{x} = \left[sf'(x^*) - \delta \right] (x(t) - x^*)$. Since $sf'(x^*) - \delta = (a-1)\delta < 0$, the FSS is stable.

6 Concluding Remarks

This paper has developed recursive intertemporal infinite horizon optimal theory for a one dimensional state variable for a continuum of spatial sites with spillovers across sites caused by state variable diffusion. We developed a local analysis of stability of optimal steady state in this setting and found a rather surprising connection between sufficient conditions for diffusion-induced instability and quantities such as Hamiltonian curvature which play key roles in global asymptotic stability analysis of infinite horizon multi sectoral recursive optimal control models. A key topic for future research is to generalize our results here to settings where the state variable is multidimensional. Another key topic for future research is to generalize our results to more general forms of diffusion, for example, diffusion governed by a kernel as in Murray (2003, Vol. II, Chapter 12). Kernels are much more general than the localized diffusion treated in this paper because flexible kernels can represent spillovers from distant sites, not just the neighboring site diffusion treated in this paper. We believe that further development of spatial optimal control models and their stability analysis will be important for understanding pattern formation caused by intertemporal spatial interactions under different market institutions, for example, rational intertemporal expectations.

For example there are papers in Semmler (2005) that compute optimal solutions for non spatial ecological management problems and other non spatial problems that are closely related to the lake problem and renewable resource problem studied in our paper. See especially the article by Grune, Kato, and Semmler which studies optimal tax theory. It would be valuable to extend their work to study such interesting issues as the interaction between spatial spillovers, discounting, diffusion-induced instability, and optimal tax design

Dasgupta and Mäler (2003) contains a collection of papers that explore nonconvex ecosystems and their management. The models include lakes, rangelands, and boreal forests. It would be valuable to generalize this type of work to incorporate spatial spillovers and study potential diffusion impacts on stability in a setting like ours. Our example of spatially connected lakes gives an indication of what such a generalization would look like, but it merely scratches the surface of potential work that could be done.

We close this paper with one more suggestion for future research which we think is very important. Magill (1977a,b) exposed the key role of curvature matrices like the Q matrix in proving local stability theorems for optimized systems in one of his early papers referenced in Magill (1977a). He also developed local linear quadratic approximations for optimized systems forced by small stochastic shocks and showed how to compute the frequency spectra for such approximations in Magill (1977b) and even showed how to apply it to business cycle analysis. Brock and Xepapadeas (2005) developed some theory of linear quadratic approximations in spatial systems which is not included in this paper due to lack of space. We believe that a very important research project would be to develop "small noise" linear quadratic approximation theory by extending Magill's (1977a,b) work to optimized spatial systems. Linear quadratic approximations are sometimes quite accurate in practical work. We believe that such approximations, as well as the general methods developed in this paper, will become very useful to economists in future applications. Indeed, we plan to work on many of these applications ourselves.

Appendix

Proof: Maximum Principle under Diffusion: Necessary Conditions (MPD-NC)

We develop a variational argument along the lines of Kamien and Schwartz (1981, pp. 115-116). Problem (1) to (5) can be written as:

$$J = \int_{z_0}^{z_1} \int_0^{\infty} e^{-\rho t} f\left(x\left(t,z\right), u\left(t,z\right)\right) dt dz = \tag{66}$$

$$\int_{z_0}^{z_1} \int_0^{\infty} e^{-\rho t} \left\{ f\left(x\left(t,z\right), u\left(t,z\right)\right) + \right. \tag{67}$$

$$\left. q\left(t,z\right) \left[g\left(x\left(t,z\right), u\left(t,z\right)\right) + D\frac{\partial^2 x}{\partial z^2} - \frac{\partial x}{\partial t} \right] \right\} dt dz \tag{68}$$

We integrate by parts the terms $e^{-\rho t} q\left(t,z\right) \frac{\partial x}{\partial t}$ and $e^{-\rho t} q\left(t,z\right) D\frac{\partial^2 x}{\partial z^2}$ of (67). The $e^{-\rho t} q\left(t,z\right) \frac{\partial x}{\partial t}$ term becomes:

$$(-1) \int_{z_0}^{z_1} \int_0^{\infty} e^{-\rho t} q\left(t,z\right) \frac{\partial x}{\partial t} dt =$$

$$- \int_{z_0}^{z_1} \left[\left[e^{-\rho t} q\left(t,z\right) x\left(t,z\right) \right]_0^T + \int_0^{\infty} x\left(t,z\right) \frac{\partial\left(e^{-\rho t} q\right)}{\partial t} dt \right] dz = \tag{69}$$

$$- \int_{z_0}^{z_1} \left[\left[e^{-\rho t} q\left(t,z\right) x\left(t,z\right) \right]_0^T + \int_0^{\infty} e^{-\rho t} x\left(t,z\right) \left(-\rho q + \frac{\partial q}{\partial t} \right) \right] dz \tag{70}$$

where $t = 0,\ t = T$ and $T \to \infty$ in the second line of the right hand side.

In (70) the first term in T goes to zero as $T \to \infty$ by the intertemporal transversality condition. The initial term at zero that is left does not impact the expression where control appears.

By using the appropriate spatial transversality conditions:

- For the circle: $q\left(t,z_1\right) = q\left(t,z_0\right)$, $\frac{\partial x(t,z_1)}{\partial z} = \frac{\partial x(t,z_0)}{\partial z}$,
- For the zero flux: $\frac{\partial x(t,z_1)}{\partial z} = \frac{\partial x(t,z_0)}{\partial z} = 0$
- For the hostile boundary: $q\left(t,z_1\right) = q\left(t,z_0\right) = 0$

the term $e^{-\rho t}q\left(t,z\right)D\frac{\partial^{2}x}{\partial z^{2}}$ becomes

$$D\int_{z_0}^{z_1}\int_0^{\infty}e^{-\rho t}q\left(t,z\right)\frac{\partial^{2}x\left(t,z\right)}{\partial z^{2}}dtdz=$$
$$D\int_0^{\infty}\left[e^{-\rho t}\left(q\left(t,z_1\right)\frac{\partial x\left(t,z_1\right)}{\partial z}-q\left(t,z_0\right)\frac{\partial x\left(t,z_0\right)}{\partial z}\right)-\int_{z_0}^{z_1}e^{-\rho t}\frac{\partial x\left(t,z\right)}{\partial z}\frac{\partial q\left(t,z\right)}{\partial t}dz\right]dt= \tag{71}$$
$$-D\int_0^{\infty}\left[\int_{z_0}^{z_1}e^{-\rho t}\frac{\partial x\left(t,z\right)}{\partial z}\frac{\partial q\left(t,z\right)}{\partial t}dz\right]dt$$

Integrating by parts once more, we obtain

$$\begin{aligned}&\left(-1\right)D\int_0^{\infty}\left[\int_{z_0}^{z_1}e^{-\rho t}\frac{\partial x\left(t,z\right)}{\partial z}\frac{\partial q\left(t,z\right)}{\partial t}dz\right]dt=\\&D\int_0^{\infty}e^{-\rho t}\left[-\frac{\partial q\left(t,z_1\right)}{\partial z}x\left(t,z_1\right)+\frac{\partial q\left(t,z_0\right)}{\partial z}x\left(t,z_0\right)\right.\\&\left.+\int_{z_0}^{z_1}x\left(t,z\right)\frac{\partial^{2}q\left(t,z\right)}{\partial z^{2}}\right]dzdt\end{aligned} \tag{72}$$

Thus (67) becomes

$$\begin{aligned}&\int_{z_0}^{z_1}\int_0^{\infty}e^{-\rho t}f\left(x\left(t,z\right),u\left(t,z\right)\right)dtdz=\\&\int_{z_0}^{z_1}\int_0^{\infty}e^{-\rho t}\left[f\left(x\left(t,z\right),u\left(t,z\right)\right)+q\left(t,z\right)g\left(x\left(t,z\right),u\left(t,z\right)\right)\right.\\&\left.+x\left(t,z\right)\left(-\rho q+\frac{\partial q}{\partial t}\right)+x\left(t,z\right)D\frac{\partial^{2}q\left(t,z\right)}{\partial z^{2}}\right]dtdz\\&-\int_{z_0}^{z_1}\left[e^{-\rho t}q\left(t,z\right)x\left(t,z\right)\right]_0^T dz+\\&D\int_0^{\infty}e^{-\rho t}\left[-\frac{\partial q\left(t,z_1\right)}{\partial z}x\left(t,z_1\right)+\frac{\partial q\left(t,z_0\right)}{\partial z}x\left(t,z_0\right)\right]dt\end{aligned} \tag{73}$$

where $t=0,\ t=T$ and $T\to\infty$ in the third line of the right hand side.

We consider a one parameter family of comparison controls, $u^{*}\left(t,z\right)+\epsilon\eta\left(t,z\right)$, where $u^{*}\left(t,z\right)$ is the optimal control, $\eta\left(t,z\right)$ is a fixed function and ϵ is a small parameter. Let $y\left(t,z,\epsilon\right)$, $t\in[0,\infty)$, $z\in\left[z_0,z_1\right]$ be the state variable generated by (2) and any of the spatial boundary conditions with control $u^{*}\left(t,z\right)+\epsilon\eta\left(t,z\right)$, $t\in[0,\infty)$, $z\in\left[z_0,z_1\right]$. We assume that $y\left(t,z,\epsilon\right)$ is a smooth function of all its arguments and that ϵ enters parametrically. For

$\epsilon = 0$, we have the optimal path $x^*(t,z)$. Furthermore all comparison paths must satisfy initial and circle, or zero flux or hostile boundary conditions. Thus,

$$
\begin{aligned}
y(t,z,0) &= x^*(t,z)\,,\ y(0,z,\epsilon) = x(0,z) \text{ fixed} && (74)\\
y(t,z_0,\epsilon) &= y(t,z_1,\epsilon) = \bar{y}(t)\,,\ \text{circle} && (75)\\
\left.\frac{\partial y(t,z,\epsilon)}{\partial z}\right|_{z=z_0} &= \left.\frac{\partial y(t,z,\epsilon)}{\partial z}\right|_{z=z_1} = 0\ ,\ \text{zero flux} && (76)\\
y(t,z_1,\epsilon) &= y(t,z_0,\epsilon) = 0,\ \text{hostile boundary} && (77)
\end{aligned}
$$

When the functions $u^*, x^*$and η are held fixed, the value of (66) evaluated along the control function $u^*(t,z) + \epsilon\eta(t,z)$ and the corresponding state function $y(t,z,\epsilon)$ depend only on the single parameter ϵ. Therefore,

$$
J(\epsilon) = \int_{z_0}^{z_1}\int_0^{\infty} e^{-\rho t}\left[f\left(y(t,z,\epsilon), u^*(t,z) + \epsilon\eta(t,z)\right)\right] dtdz
$$

or using (78)

$$
\begin{aligned}
&J(\epsilon) = \\
&\int_{z_0}^{z_1}\int_0^{\infty} e^{-\rho t}\left[f\left(y(t,z,\epsilon), u^*(t,z) + \epsilon\eta(t,z)\right)\right.\\
&+ q(t,z)\,g\left(y(t,z,\epsilon), u^*(t,z) + \epsilon\eta(t,z)\right)\\
&\left.+y(t,z,\epsilon)\left(-\rho q(t,z) + \frac{\partial q(t,z)}{\partial t}\right) + Dy(t,z,\epsilon)\frac{\partial^2 q(t,z)}{\partial z^2}\right] dtdz\\
&- \int_{z_0}^{z_1} e^{-\rho t}\left[q(t,z)\,y(t,z,\epsilon)\right]_0^T dz\\
&+ D\int_0^{\infty} e^{-\rho t}\left[-\frac{\partial q(z_1)}{\partial z}y(t,z_1,\epsilon) + \frac{\partial q(z_0)}{\partial z}y(t,z_0,\epsilon)\right] dt \qquad (78)
\end{aligned}
$$

where $t = 0,\ t = T$ and $T \to \infty$ in the third line of the right hand side. (79)

Since u^* is a maximizing control, the function $J(\epsilon)$ assumes a maximum when $\epsilon = 0$. Thus $\left.\frac{dJ(\epsilon)}{d\epsilon}\right|_{\epsilon=0} = 0$ or

$$\left.\frac{dJ(\epsilon)}{d\epsilon}\right|_{\epsilon=0} =$$
$$\int_{z_0}^{z_1}\int_0^{\infty}\left[e^{-\rho t}\left[\left(f_x + qg_x + \frac{\partial q}{\partial t} - \rho q(t,z) + D\frac{\partial^2 q}{\partial z^2}\right)y_\epsilon + (f_u + qg_u)\,\eta(t,z)\right]\right]dtdz +$$
$$-\int_{z_0}^{z_1}\left[e^{-\rho t}q(t,z)\,y_\epsilon(t,z,0)\right]_0^T dz +$$
$$D\int_0^{\infty} e^{-\rho t}\left[-\frac{\partial q(z_1)}{\partial z}y_\epsilon(t,z_1,0) + \frac{\partial q(z_0)}{\partial z}y_\epsilon(t,z_0,0)\right]dt = 0 \tag{80}$$

where $t = 0, t = T$ and $T \to \infty$ in the second line of the right hand side.

In (80), $y_\epsilon(0,z,\epsilon) = 0$, since $y(0,z,\epsilon) = x(0,z)$ fixed by initial conditions. Using this we show next, by assuming an appropriate transversality condition at infinity when the state and costate variables are positive and the state variable is bounded away from zero, that the last term of (80) for each type of spatial transversality conditions vanishes.

- Circle: Let
$$\int_{z_0}^{z_1}\xi(T,z)\,\beta(T,z) = 0 \tag{81}$$
for all $\beta(T,z)$ piecewise continuous functions in $[z_0, z_1]$. It follows, using Athans and Falb's (1966, p. 260) fundamental lemma, that
$$\xi(T,z) = 0\ ,\ z \in [z_0, z_1] \tag{82}$$
By writing $\xi(T,z) = e^{-\rho T}q(T,z)$ and assuming the intertemporal transversality condition $\lim_{T\to\infty} e^{-\rho T}q(T,z) = 0$, we obtain
$$\lim_{T\to\infty} e^{-\rho T}\int_{z_0}^{z_1} e^{-\rho T}q(T,z)\,dz = 0 \tag{83}$$
$$\lim_{T\to\infty} e^{-\rho T}\int_{z_0}^{z_1} q(T,z)\,x(T,z)\,dz = 0 \text{ or,} \tag{84}$$
$$\lim_{T\to\infty} e^{-\rho T}q(T,z) = 0\ ,\ \lim_{T\to\infty} e^{-\rho T}q(T,z)\,x(T,z) = 0 \tag{85}$$
Then since $y_\epsilon(t,z,\epsilon)$ is arbitrary
$$-\left[e^{-\rho t}q(t,z)\,y_\epsilon(t,z,\epsilon)\right]_0^T = 0,\ T \to \infty$$
Using $y(t,z_1,0) = y(t,z_0,0) = \bar{y}(t)$, assuming $y_\epsilon(t,z_1,0) = y_\epsilon(t,z_0,0)$

as it is appropriate from the circle conditions $x(t,z_0) = x(t,z_1) = \bar{x}(t)$, $\frac{\partial x(t,z_0)}{\partial z} = \frac{\partial x(t,z_1)}{\partial z}$ and using the spatial transversality conditions $\frac{\partial q(t,z_1)}{\partial z} = \frac{\partial q(t,z_0)}{\partial z}$, we obtain

$$\left[-\frac{\partial q(t,z_1)}{\partial z} y_\epsilon(t,z_1,0) + \frac{\partial q(t,z_0)}{\partial z} y_\epsilon(t,z_0,0)\right] = 0$$

- Zero flux: Using as above $\lim_{T\to\infty} e^{-\rho T} q(T,z) = 0$ and the zero flux spatial transversality conditions on q,

$$\frac{\partial q(t,z_1)}{\partial z} = \frac{\partial q(t,z_0)}{\partial z} = 0 \tag{86}$$

  the last two terms of the right hand side of (80) vanish again.

- Hostile Boundary: Using $\lim_{T\to\infty} e^{-\rho T} q(T,z)$ and $y_\epsilon(t,z_1,0) = y_\epsilon(t,z_0,0) = 0$ since $y(t,z_1,\epsilon) = y(t,z_0,\epsilon) = 0$ fixed, by the hostile boundary assumption, the last two terms of the right hand side of (80) vanish again.

Since y_ϵ and $\eta(t,z)$ are arbitrary, we obtain from (80) that necessary conditions for a local maximum are:

$$\frac{\partial q}{\partial t} = \rho q - \left(f_x + q g_x + D\frac{\partial^2 q}{\partial z^2}\right) \tag{87}$$

$$f_u + q g_u = 0 \tag{88}$$

So if we define a generalized current value Hamiltonian function

$$\tilde{H} = f(x,u) + q\left[g(x,u) + D\frac{\partial^2 x}{\partial z^2}\right]$$

then by (87) and (88) we obtain (7)-(9) along with the appropriate spatial transversality conditions.■

Maximum Principle under Diffusion: Sufficient conditions (MPD-SC)

Assume that functions $f(x,u)$ and $g(x,u)$ are concave differentiable functions for problem (1)-(5) and suppose that functions $x^*(t,z), u^*(t,z)$ and $q(t,z)$ satisfy necessary conditions (7)-(9) for all $t \in [0,\infty)$, $z \in [z_0,z_1]$ and that $x(t,z)$ and $q(t,z)$ are continuous with

$$q(t,z) \geq 0 \text{ for all } t \text{ and } z. \tag{89}$$

Then the functions $x^*(t,z), u^*(t,z)$ solve the problem (1)-(5). That is, the necessary conditions (7)-(9) are also sufficient

The result can easily be extended along the lines of the Arrow and Kurz sufficiency theorem (Arrow and Kurz, 1970, Chapter II.6) with the transversality condition at infinity:

$$\lim_{t\to\infty} e^{-\rho t}\int_{z_0}^{z_1} q(t,z)\,dz \geq 0,\ \lim_{t\to\infty} e^{-\rho t}\int_{z_0}^{z_1} q(t,z)\,x(t,z)\,dz = 0, \text{or} \tag{90}$$

$$\lim_{t\to\infty} e^{-\rho t} q(t,z)\,x(t,z) = 0 \text{ when } (q(t,z), x(t,z)) \geq 0 \text{ for all } t,z \tag{91}$$

Proof

Suppose that $x^*(t,z), u^*(t,z), q(t,z)$ satisfy conditions (7) and (8) and let $x(t,z), u(t,z)$ functions satisfy (2). Let f^*, g^* denote functions evaluated along $(x^*(t,z), u^*(t,z))$ and let f, g denote functions evaluated along the feasible path $(x(t,z), u(t,z))$. To prove sufficiency we need to show that

$$W \equiv \int_{z_0}^{z_1}\int_0^{\infty} e^{-\rho t}(f^*-f)\,dtdz \geq 0$$

From the concavity of f it follows that

$$f^* - f \geq (x^*(t,z) - x(t,z)) f_x^* + (u^*(t,z) - u(t,z)) f_u^* \tag{92}$$

Then

$$W \geq \int_{z_0}^{z_1}\int_0^{\infty} e^{-\rho t}\left[(x^*(t,z)-x(t,z)) f_x^* + (u^*(t,z)-u(t,z)) f_u^*\right] dtdz \tag{93}$$

$$= \int_{z_0}^{z_1}\int_0^{\infty} e^{-\rho t}\left[(x^*(t,z)-x(t,z))\left(-\frac{\partial q(t,z)}{\partial t} - q(t,z)\,g_x^* - D\frac{\partial^2 q(t,z)}{\partial z^2} + \rho q(t,z)\right)\right. \tag{94}$$

$$\left. + (u^*(t,z)-u(t,z))(-q(t,z)\,g_u^*)\right] dtdz$$

$$= \int_{z_0}^{z_1}\int_0^{\infty} e^{-\rho t} q\left[(g^*-g) - (x^*(t,z)-x(t,z))\,g_x^* - (u^*(t,z)-u(t,z))\,g_u^*\right] dtdz \geq 0 \tag{95}$$

Condition (94) follows from (93) by using conditions (7) and (8) to substitute for f_u^* and f_x^*. Condition (95) is derived by using (70) to replace the term $e^{-\rho t}(x^*-x)(-\partial q/\partial t + \rho q)$, (71)-(72) to replace the term $De^{-\rho t}(x^*-x)(\partial^2 q/\partial z^2)$, (2) to replace the term $\partial x/\partial t$ that enters the expression after the substitutions, and the appropriate intertemporal and spatial transversality conditions. The non-negativity of the integral in (95) follows from (89) and the concavity of g.■

The Arrow and Kurz type sufficiency theorem follows from the concavity of f and g since $\max_u H(x,u,q)$ is concave in x for any given $q \geq 0$.

Arrow and Kurz type sufficiency condition: *Let H^0 denote the maximized Hamiltonian, or $H^0(x,q) = \max_u H(x,u,q)$. If the maximized Hamiltonian is a concave function of x for given q, then functions $x^*(t,z)$, $u^*(t,z)$ and $q(t,z)$ that satisfy conditions (7)-(9), the appropriate spatial transversality conditions and the transversality condition at infinity (90), (91) solve the problem (1)-(5).*

Proof: The proof of this result closely follows the standard proofs available in the literature.

Proof of Theorem 1

Define deviations from the FOSS $\mathbf{w} = (x(t,z) - x^*, q(t,z) - q^*)$, denote partial derivatives by subscripts and write the linearization of the full MHDS

$$\mathbf{w}_t = J^0\mathbf{w}+\tilde{D}\mathbf{w}_{zz}\,,\, \tilde{D} = \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix} \tag{96}$$

Following Murray (2003) we consider the time-independent solution of the spatial eigenvalue problem, with appropriate boundary conditions

$$\mathbf{W}_{zz} + k^2\mathbf{W}=0 \tag{97}$$

where k is the eigenvalue. For the one-dimensional domain $[0,L]$ we have solutions for (97) which are of the form

$$\mathbf{W}_k(z) = \mathbf{A}_n \cos\left(\frac{2n\pi z}{L}\right) \,,\; n = \pm 1,\ \pm 2, ..., \tag{98}$$

where $\mathbf{A}_n$ are arbitrary constants. Solution (98) satisfies circle boundary conditions at $z = 0$ and $z = L$. The eigenvalue is $k = 2n\pi/L$, and $1/k = L/2n\pi$ is a measure of the wave-like pattern. The eigenvalue k is called the *wavenumber* and $1/k$ is proportional to the wavelength $\omega : \omega = 2\pi/k = L/n$. Let $\mathbf{W}_k(z)$ be the eigenfunction corresponding to the wavenumber k. We then look for solutions to (96) of the form

$$\mathbf{w}(t,z) = \sum_k c_k e^{\lambda t}\mathbf{W}_k(z) \tag{99}$$

Substituting (99) into (96), using (97) and canceling $e^{\lambda t}$, we obtain for each k or equivalently each n, that $\lambda\mathbf{W}_k = J^0\mathbf{W}_k - Dk^2\mathbf{W}_k$. Since we require non-trivial solutions for $\mathbf{W}_k$, λ must solve

$$\left|\lambda I - J^0 + \tilde{D}k^2\right| = 0$$

Then the eigenvalue $\lambda(k)$ as a function of the wavenumber is obtained as the roots of

$$\lambda^2 - \rho\lambda + h\left(k^2\right) = 0 \tag{100}$$

$$h\left(k^2\right) = -D^2k^4 + D\left(2H^0_{xq} - \rho\right)k^2 + \det J^0 \tag{101}$$

where the roots are given by:

$$\lambda_{1,2}\left(k^2\right) = \frac{1}{2}\left(\rho \pm \sqrt{\rho^2 - 4h\left(k^2\right)}\right)$$

It should be noted that the flat $(D = 0)$ case corresponds to $k^2 = 0$, so that $h\left(k^2 = 0\right) = \det J^0$, and $\lambda_{1,2} = \frac{1}{2}\left(\rho \pm \sqrt{\rho^2 - 4\det J^0}\right)$. In this case Kurz's result holds, either $(\lambda_2, \lambda_1) > 0$ and the FOSS is unstable or $\lambda_2 < 0 < \lambda_1$ and the FOSS is saddle point stable, and there is a one-dimensional stable manifold containing the FOSS. Solutions of the MHDS on this manifold are GAS.

We consider now the impact of a perturbation induced by diffusion on a saddle point FOSS. Under diffusion the smallest root λ_2 is given by

$$\lambda_2\left(k^2\right) = \frac{1}{2}\left(\rho - \sqrt{\rho^2 - 4h\left(k^2\right)}\right) \tag{102}$$

Then,

- If $0 < h\left(k^2\right) < \rho^2/4$ for some k, then λ_2 becomes real and positive.
- If $h\left(k^2\right) > \rho^2/4$ for some k, then both roots corresponding to λ are complex with positive real parts.

In both cases above, the linearly stable steady state, on the stable manifold, becomes unstable to *spatial* disturbances. Therefore if $h\left(k^2\right) > 0$ for some k, then $\lambda_2\left(k^2\right) > 0$ and the optimally controlled Hamiltonian system becomes unstable to spatial perturbations, in the neighborhood of the flat steady state and along the stable manifold. From (101) the quadratic function $h\left(k^2\right)$ is concave, and therefore has a maximum. Furthermore, $h(0) = \det J^0 < 0$ and $h'(0) = \left(2H^0_{xq} - \rho\right)$. Then $h\left(k^2\right)$ has a maximum for

$$k_{\max}^{2} : h^{'}\left(k_{\max}^{2}\right) = 0, \text{ or } k_{\max}^{2} = \frac{\left(2H_{xq}^{0} - \rho\right)}{2D} > 0, \text{ for } \left(2H_{xq}^{0} - \rho\right) > 0 \quad (103)$$

If $h\left(k_{\max}^{2}\right) > 0$ or $-D^{2}k_{\max}^{4} + D\left(2H_{xq}^{0} - \rho\right)k_{\max}^{2} + \det J^{0} > 0$, and $2H_{xq}^{0} - \rho > 0$, then there exist two positive roots $k_{1}^{2} < k_{2}^{2}$ such that $h\left(k^{2}\right) > 0$ and $\lambda_{2}\left(k^{2}\right) > 0$ for $k^{2} \in \left(k_{1}^{2}, k_{2}^{2}\right)$. Using (103) the existence of two positive roots $k_{1}^{2} < k_{2}^{2}$ requires

$$\frac{\left(2H_{xq}^{0} - \rho\right)^{2}}{4} + \det J^{0} > 0 \quad (104)$$

$$\text{which is equivalent to } \frac{\rho^{2}}{4} > -H_{xx}^{0}H_{qq}^{0} \quad (105)$$

$$k_{1,2}^{2} = \frac{\left(2H_{xq}^{0} - \rho\right) \pm \sqrt{\left(\rho^{2} + 4\left(H_{xx}^{0}H_{qq}^{0}\right)\right)}}{2D} > 0 \quad (106)$$

The interval (k_{1}, k_{2}) determines the range of the unstable modes associated with the spatial heterogeneous solution, while $h\left(k^{2}\right)$ is the dispersion relationship associated with the optimal control problem. Diffusion driven instability in the optimally controlled system emerges if the maximum of the dispersion relationship is in the positive quadrant. These conditions are summarized below.

$$2H_{xq}^{0}\left(x^{*}, q^{*}\right) > \rho \quad (107)$$

$$\frac{\rho^{2}}{4} > -H_{xx}^{0}\left(x^{*}, q^{*}\right)H_{qq}^{0}\left(x^{*}, q^{*}\right) \quad (108)$$

■

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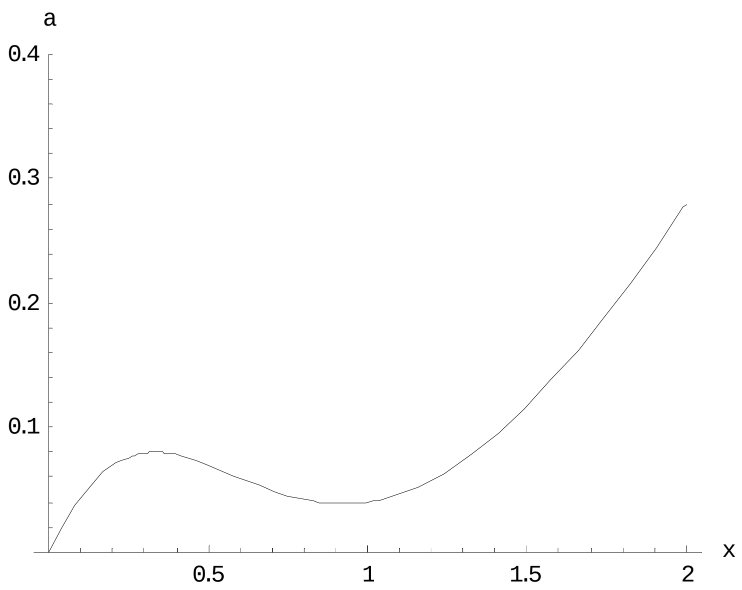

Figure 1: Shallow lake: Reversibility and hysteresis

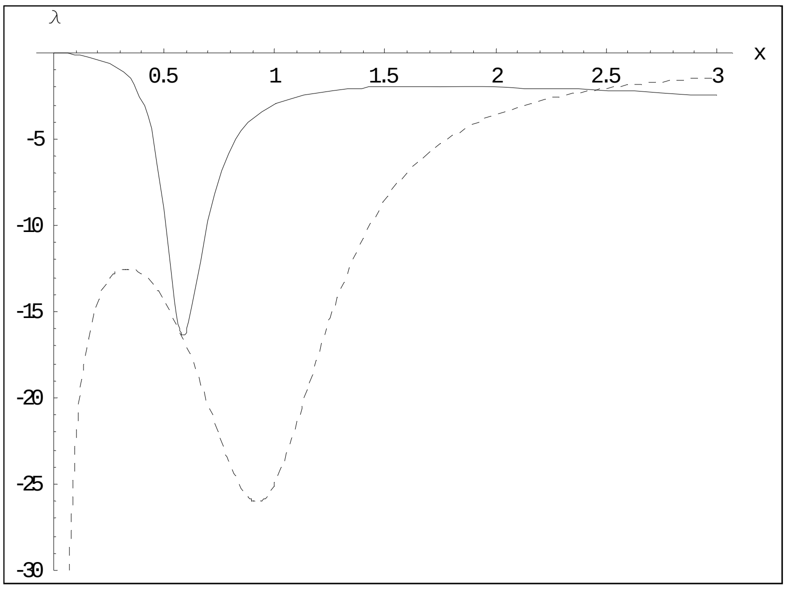

Figure 2: Flat optimal steady states for the shallow lake.

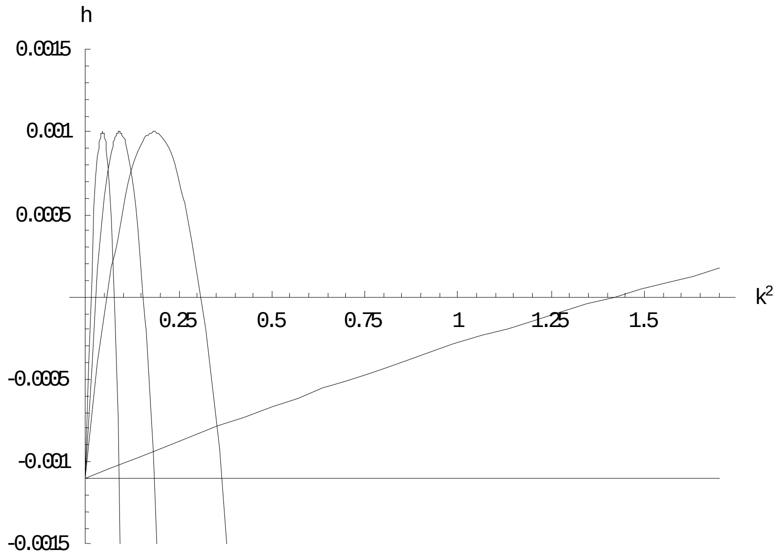

Figure 3: The dispersion relationship

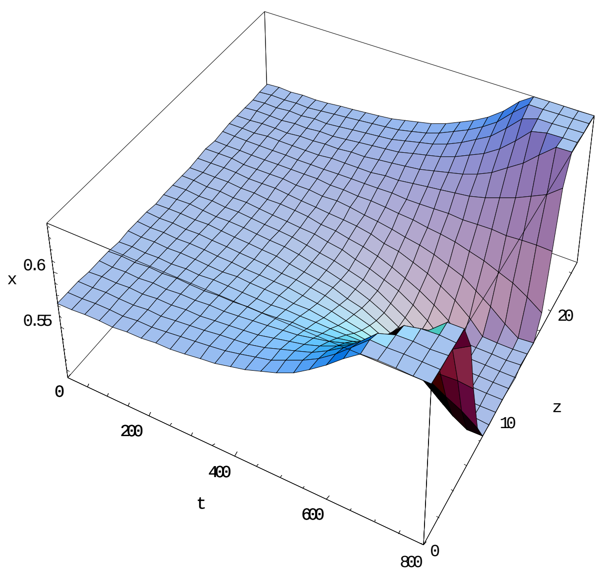

Figure 4: The growing spatial instability

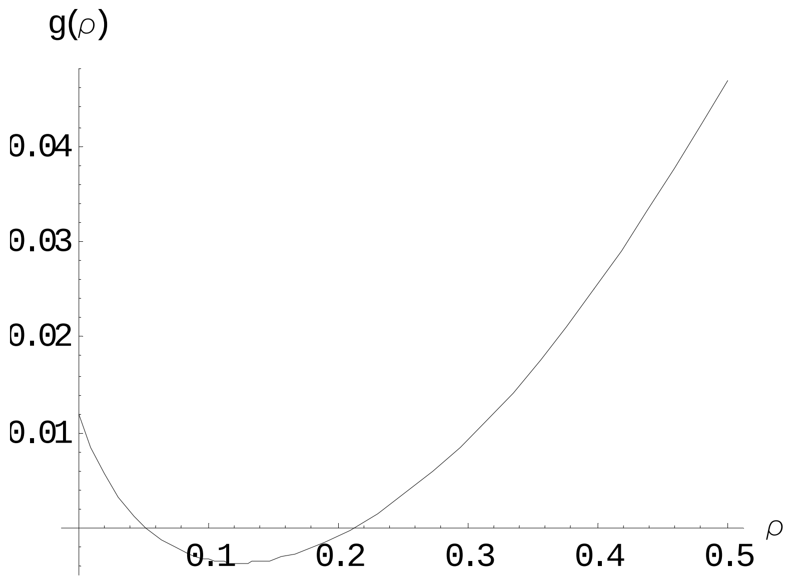

Figure 5: The $g(\rho)$ relationship: Optimal resource management with stock effects

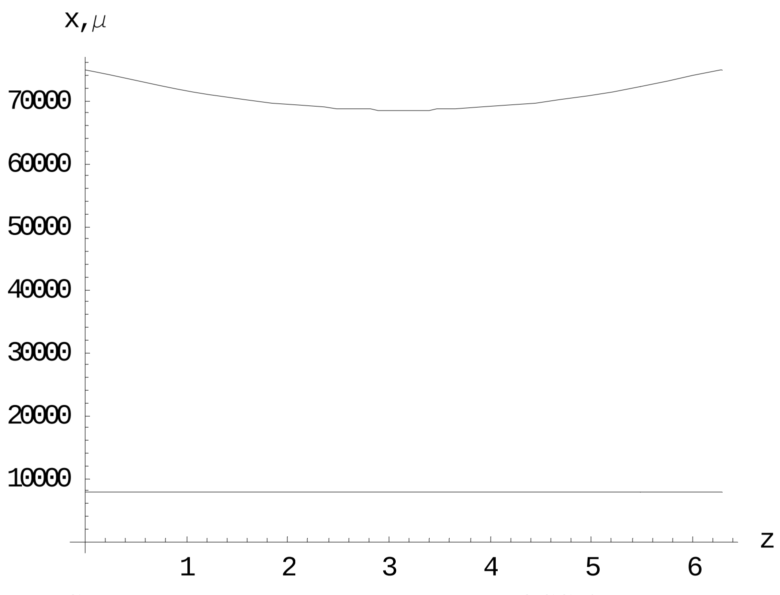

Figure 6: Spatial biomass pattern at the HOSS for optimal resource management with stock effects