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Poverty traps and economic growth in a two-sector model of subsistence agriculture

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Abstract

This paper demonstrates how a self-reinforcing poverty trap can arise within a standard macroeconomic growth model, solely by recognizing the fact that some goods, such as for example food, are essential to human life. This is done by studying qualitative model dynamics within a two-sector growth model of subsistence agriculture. The two main findings are; 1) the introduction of subsistence food requirements in a two sector framework leads to the emergence of poverty trap like features. 2) this also gives rise to an indifference point where the decision maker is indifferent in choosing between two equally optimal solutions.

These results are of importance for a wide branch of economic growth papers that have been exploring how subsistence agriculture impacts the transition path to modern economic growth. Our results suggests that when preferences are non-homothetic, as is the case when a subsistence requirement is introduced, this effectively restricts the amount of admissible solutions. Further, the existence of an indifference points also poses a question as to whether the calculated optimal paths, if admissible, are actually maximizing social welfare or are just second best alternatives.

Keywords: two-sector, poverty trap, indifference point, subsistence

1 Introduction

This paper discusses, in the context of economic growth models, the implications of distinguishing between goods that are easily substitutable from those which are essential for human well-being. The importance of this distinction becomes clear first when subsistence levels are approached and has therefore become intensively studied within the field of development economics. Empirical findings within this field suggests that the lack of development in many poor

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countries is due to the fact that a very high proportion of income is devoted to food production.[1] Theodore Schultz characterized this as the "food problem", in which a country must first be able to meet its own subsistence needs before the transition to modern economic growth can start to take off. This particular feature of agricultural production, in combination with Engel's law[2], has given rise to a particular paradox in the literature on agricultural development (Timmer, 2009). The paradox lies in the observation, that for a poor country to transition out of poverty, major investments in agriculture are needed, but when economic growth later starts to take off, the relative importance of the agricultural sector will start to decline.

A common way that the economic growth literature has tried to capture the subsistence feature of agricultural production in growth models has been through the inclusion of a subsistence parameter that addresses the minimum consumption requirement of agricultural goods below which individuals cannot survive.[3] This way of expressing subsistence consumption is definitely not new and has previously been explored in models of exhaustible resources (Koopmans, 1973, 1974; Vousden, 1973). Their findings, show that when a minimum consumption requirement is introduced into such a model this bears consequences for its qualitative properties. In particular, they show that under such a constraint a solution having an infinite time horizon may not exist and the time horizon could therefore be considered as a choice variable along with other variables such as the discount rate.[4]

In this paper we explore how the results of Koopmans and Vousden carry over to economic growth models of subsistence agriculture. In particular we focus on exploring the role played by the subsistence parameter and its significance for global model dynamics. We show that the results found in exhaustible resource models also hold for the case where the exhaustible resource stock is exchanged for a producible commodity such as agricultural goods.[5] The results are further complicated due to the fact that we allow for finite time horizon solutions even when solution paths having infinite time horizons also exist. In our model, it is thus not only quantity of life that matters but also quality of life. This also implies that there must exist a threshold value where the optimal solution switches from having a finite time horizon to an infinite time horizon. We provide a complete analytical proof that such a threshold exists and comes in the form of an indifference point.[6]

[1]See for example: Schultz (1953), Lewis (1955), Johnston and Mellor (1961), Kuznet (1966), Timmer (1988)

[2]Engel's law states that as incomes rise the share of food in consumer spending declines.

[3]Some important contributions that have modeled subsistence consumption in this way include that of Matsuyama (1992), King and Rebelo (1993), Echevarria (1997), Galor and Weil (2000), Steger (2000), Kongsamut et al. (2001), Gollin et al. (2002, 2007), Irz and Roe (2005) and Kraay and Raddatz (2007).

[4]See also Dasgupta (1978) for a general discussion.

[5]We use agricultural goods as a proxy for food production in general and it could therefore be seen as also including production from fisheries or any other unbearable consumption good

[6]Following the terminology of Wagener (2006) and Kiseleva and Wagener (2010), the term indifference point refers to what has in previous literature also become known as a Skiba or DNSS point.

The results found here will thus have several implications for empirical growth models of poor or malnourished countries. For example, if an infinite time horizon is imposed as an a priori assumption, these models could be missing optimal candidate solutions having finite planning horizons, which would have yielded a higher overall welfare than their respective counterparts. If the empirical estimates of initial resources hence lie close to subsistence levels this would then become a problem. Further, if uncertainties concerning initial states or parameter estimates are significant, the calculated optimal solution paths could also change dramatically. Letting the planning horizon become an endogenous decision variable in this way will thus imply that initial resources or wealth will affect the way we plan and discount for the future.

This paper is organized as follows. Section 2 presents the main features of our model and discusses how poverty traps emerge within its context. Section 3 introduces the complete model description. Section 4 presents the analytical results of importance. Section 5 provides some more specific numerical results. Section 6 discusses the results and possible extensions for future research.

2 Economic growth and poverty traps

This section introduces a two sector model of an economy where one of the sectors is essential for life support. We show how poverty traps can emerge as a direct consequence of the assumptions made and study global system dynamics to assess all possible outcomes of the model. The model built here is a minimal conceptual model that could represent the dynamics of growth for a made-up global economy where subsistence requirements are a potential issue alternatively for a poor isolated country. Some of the future scenarios described by IPCC (2007) indicate that there is at least a possibility that such subsistence issues could become important again, even at the global level. In addition subsistence agriculture is still an issue for many developing countries where domestic agricultural production is of great importance for economic growth (World Bank, 2008). Our model depicts the story of a benevolent social planner trying to achieve a maximum amount of social well-being for its citizens. In doing so the planner must consider the social preferences of its citizens, and how they value not only agricultural versus manufacturing goods, but also present consumption against future consumption by themselves or their descendants.

2.1 Production in two sectors

Consider a stylized economy consisting of two sectors, manufacturing (m) and agricultural or food producing sector (a)[7], that produce two homogeneous goods. The manufactured good can be used for either consumption or investment, whilst the life-supporting good is a pure consumption good. Production technologies in each sector are taken to be of the Cobb-Douglas form, where the

[7] The life supporting good could be food, or any other unbearable consumption good.

relative factor share of capital $K(t)$ and labor $L(t)$ used in each sector determines aggregate production in each sector and suffixes m and a characterize each sector. Here, $Y_m(t)$ and $Y_a(t)$ denote levels of output and A_m and A_a, are constants describing sector specific productivity differences, while α and β determine the factor shares used in production.

$$Y_m(t) = A_m K_m(t)^\alpha L_m(t)^{1-\alpha}$$
$$Y_a(t) = A_a K_a(t)^\beta L_a(t)^{1-\beta}$$

We can show that the qualitative results are not contingent on keeping labor as a time dependent variable in the model (see Appendix A). For ease of exposition labor is therefore held constant in each sector. We thus rewrite the production function in per capita terms, without loss of generality :

$$y_m(t) = A_m k_m(t)^\alpha \tag{1a}$$
$$y_a(t) = A_a k_a(t)^\beta \tag{1b}$$

where $y_m := Y_m/L_m,\ y_a := Y_a/L_a$ and $k_m := K_m/L_m,\ k_a := K_a/L_a$. Further it is assumed that there are no market frictions so that capital can move freely and costlessly between the two sectors and that markets clear at each moment in time so that:

$$k(t) = k_m(t) + k_a(t) \tag{2}$$

By assumption, agricultural production is directly consumed, so a minimum subsistence requirement directly translates into a minimum production constraint per capita for the agricultural sector. Let γ denote a subsistence level of agricultural production:[8]

$$y_a(t) = A_a k_a(t)^\beta \geq \gamma \tag{3}$$

The parameter γ corresponds to the concept of a poverty line used by the World Bank, which identifies the part of a population, which is to be regarded as absolutely poor.[9] Letting $c_m(t)$ denote per capita consumption of the manufactured good and δ capital depreciation we proceed and write the capital budget flow equation as

$$\dot{k}(t) = A_m k_m(t)^\alpha - \delta k(t) - c_m(t) \tag{4}$$

Making use of the market clearing condition (2) and the subsistence food requirement (3) it is already possible to say at least two things regarding the characteristics of the budget constraint (4). First, for a given positive level of subsistence ($\gamma > 0$) there exists initial per capita capital stocks $k_0 > 0$ for which the subsistence requirement (3) is not satisfied for any feasible choice of k_m. In Prop. A.1 we show that this is true for all initial capital stocks lying in the region $0 \leq k < \gamma^{\frac{1}{\beta}}$. This implies that the subsistence requirement (3) effectively

[8] Per capita variables will hereafter be denoted by lower case letters.

[9] The World Bank (2008) reports two such poverty lines, one lower level corresponding to 1.08$ per day and also an upper poverty line of 2.15$ a day, defined in 1993 purchasing power parity dollars.

restricts the state space of the model. Second, from this proposition it also follows that for a positive γ there will exist a per capita capital level $k > \gamma^{\frac{1}{\beta}}$ for which the capital dynamics will become negative regardless of how the levels of per capita consumption, $c_m(t)$, and capital, $k_m(t)$, in the manufacturing sector are chosen.

Figure 1 shows the different possibilities, portrayed as regions in the $k - \gamma$ space. Region I (above the dotted line) represents the region where the subsistence constraint (3) has been violated. This region lies outside the boundaries of our model. Along the dashed line the maximum value of the capital dynamics (4) are zero, meaning that given k and γ any other choices of consumption level would imply that $\dot{k} < 0$ (See Appendix A). In region II (between the dotted and the dashed line) the capital dynamics (3) are always negative $\dot{k} < 0$ regardless of how consumption and the capital stocks are chosen. In this area it is thus impossible to attain some long run equilibrium having a value of capital per capita k larger than the initial starting value. Finally, region III (below the dashed line) represents a region where the capital dynamics can be either negative or positive and where the solution always converges to some positive equilibrium in the long run. The analysis so far thus implies that there will exist no path starting in region II, which will converge to some equilibrium lying in region III.

In addition, for paths starting in region II, there will exist a finite time horizon $T < \infty$ at which the constraint (3) will become active before the capital stock reaches an equilibrium ($\dot{k} = 0$) at an infinite time horizon.

2.2 The emergence of poverty traps

Koopmans (1973, 1974) and Vousden (1973) considered problems of optimal consumption of an exhaustible finite resource stock over time. In particular these papers deal with the requirement that a minimum amount of the resource is assumed to be essential to life, in such a way that all life ceases upon its exhaustion. It is shown that this requirement implies that life cannot be extended over an infinite time horizon.

What is shown is thus that when a resource is essential to life, stocks are finite and no alternative exists, doomsday is inevitable. Although the situation is highly stylized it still captures well, how finite resources and minimum requirements might affect optimal decision making.

What about the case where the resource is essential but not necessarily finite? The economy described in section (2.1) describes such a situation. Here agricultural food production is considered to be an essential resource. A minimum life sustaining consumption quantity is given by the parameter γ. In such a situation it is shown that if the economy is not endowed with enough resources to sustain such a minimum production level over time, the time horizon will be restricted (as it was in the analysis by Koopmans and Vousden). This situation is given by region II of figure 1, here there will exist a finite time horizon $T < \infty$ at which the constraint (3) will become active. In such a situation the decision maker is trapped, thereby bearing resemblance to the notion of a poverty trap.

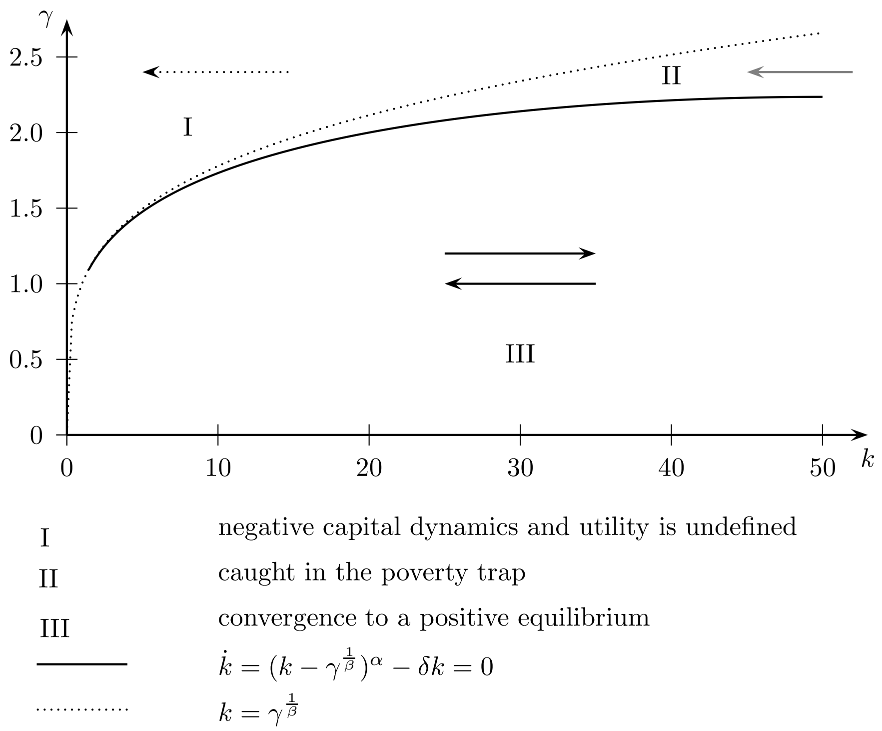


Figure 1: In this figure the different regions of dynamic behavior, depending on the initial state k_0 and actual value of γ, is depicted. The other parameter values are taken from Tab. 1.

This raises the question of how the time horizon T should be chosen. One way, is to just choose some arbitrary time horizon shorter than infinity. However, one could also imagine that the society incorporates the time horizon T as a control variable. The reason for doing so being that manipulating the time horizon could generate welfare gains under some specific circumstances.[10] The formulation as a free end time problem is actually necessary from a mathematical viewpoint because the subsistence constraint together with the capital dynamics may not be satisfied on an infinite time interval. The intuition behind this mathematical technicality is challenging though. In the real world, countries lying in region II would most likely be kept away from region I by means of foreign aid and investment entering the country. However, it might also be so that even though a foreign donation of capital could take the country out of region II and into region III, it might still not remain there, due to for example malfunctioning institutions or a corrupt government. Meanwhile our ambition to gain insights about some specific aspects of real world dynamics by analyzing a simplified representation of it rather giving an exact model of the real world.

Considering the time horizon as a control variable can actually help us gain insights on real world dynamics. One way of interpreting the choice of a finite time horizon T could be that after this time horizon has passed, the present generation is no longer willing to sacrifice present day consumption for the benefit of future generations that might come into existence after time T. This would be equivalent to setting an infinitely high discount rate after time T. This action might not be ethically defensible for a rich society, but probably more so if they were living on the brink of subsistence. Such a poor population trying to survive day-by-day might be unable or even unwilling to sacrifice their scarce amount of resources for the sake of future generations. Due to the risk and uncertainties involving the future, there might be several reasons why lack of saving, might be logical and perhaps not even ethically questionable. Is it ethically defensible to accuse a starving man for not saving enough? A rich population on the other hand cannot on similar grounds justify, why they should not sacrifice, at least some arbitrary small amount of their accumulating resources on behalf of future generations. At least, the possibility is open to them.

Thus we consider a situation when society can optimize not only consumption over time but also their planning horizon T optimally. Under this setting it is shown that rich societies will in general choose an infinite time horizon, while poor societies will choose a finite time horizon. When society is poor and no feasible investment plan exists that could alleviate poverty, the problem thus becomes technically similar to that of a cake-eating problem of optimal resource extraction. As shown in Kumar and Naqib (1993) and confirmed in our analysis, for problems of this type there might not exist solutions to the infinite time horizon problem for all initial stock sizes greater than zero.

[10] This is also considered by Koopmans (1973, 1974) or Vousden (1973)

3 Model description

We now consider the social planner's problem of maximizing social welfare. In accordance with the above analysis the planner takes into account the initial conditions of the current generation when making his planning decision. This implies that if a country is sufficiently poor (i.e. endowed with a small capital stock) the optimal planning decision might not involve converging to some positive equilibrium in the long run.

$$\max_{c_m(\cdot),k_m(\cdot),T} \int_0^T \mathrm{e}^{-\rho t}\, U(c_m(t), c_a(t)-\gamma)\ \mathrm{d}t + \mathrm{e}^{-\rho T}\, S(k(T),\gamma) \tag{5a}$$

$$\text{s.t.}\quad \dot{k}(t) = A_m k_m(t)^\alpha - \delta k(t) - c_m(t) \quad t\in[0,T] \tag{5b}$$

$$\text{with}\quad c_a(t) = A_a k_a(t)^\beta \quad t\in[0,T] \tag{5c}$$

$$k(t) = k_m(t) + k_a(t) \quad t\in[0,T] \tag{5d}$$

$$c_a(t) - \gamma \geq 0 \quad t\in[0,T] \tag{5e}$$

$$A_m k_m(t)^\alpha - c_m(t) \geq 0 \quad t\in[0,T] \tag{5f}$$

$$c_m(t), k_m(t), k_a(t) \geq 0 \quad t\in[0,T] \tag{5g}$$

For a representative household, utility is thus derived from consumption of manufacturing goods and agricultural goods; manufacturing consumption enters explicitly in the utility function whereas agricultural consumption is implicitly given by its production ($c_a(t) = A_a k_a(t)^\beta$). The constraint (5e) corresponds to the subsistence requirement that agricultural consumption must reach at least some positive value γ, i.e., a level of necessary consumption needed in order to sustain life. The constraint (5f) is a non-negativity constraint on investment. The objective function is divided into two parts where the first integral from 0 to T denotes utility from a production based economy up to a switching time T after which utility is given by the scrap value $\mathrm{e}^{-\rho T}\, S(k(T),\gamma)$.

The scrap value can in a stylized sense be interpreted as the value of abandoning the previous production technology and switching to some alternative technology or perhaps migrating to some other region. The size of the scrap value $S(k(T),\gamma)$ will depend upon the size of the capital stock at the switching time T as well as the exogenously given opportunities, which will be included as a constant parameter in the scrap function. The switching time is thus a choice variable, where switching before poverty becomes too severe will generate a higher value of migration. In the case where no alternatives exists the lack of production possibilities and the minimum subsistence requirement imply a doomsday like scenario. This can be captured by setting the scrap value equal to zero.[11]

The problem for the social planner is thus to choose not only the consumption of manufacturing good $c_m(\cdot)$ but also how capital should be allocated between

[11] For further discussions regarding such the interpretations see for example (Dasgupta, 1978; Koopmans, 1977; Withagen, 1981).

the two sectors (by choosing $k_m(\cdot)$) in order to maintain a subsistence food requirement. Apart from these usual controls, the planner must also consider an optimal switching time T^* for the planning horizon. In Prop. 1 it is shown that for small positive values of γ the choice T^* that maximizes (5a) depends upon the initial state k_0, where sufficiently low values of k_0 imply that $T^* < \infty$ and for sufficiently large k_0 we have that $T^* = \infty$. This result leads us directly to the formal definition of a poverty trap in our model:

Definition 1 (Poverty Trap) An initial state k_0 is called a *poverty trap*, if for all controls satisfying the constraints the corresponding optimal solution has a switching time $T^* < \infty$. □

Azariadis and Stachurski (2005) define poverty traps as *"any self-reinforcing mechanism which causes poverty to persist."* Although definition (1) does not explicitly say anything about a self-reinforcing mechanism, we will see that such a mechanism is an implicit result arising within our model when the initial state is a poverty trap. Let us now analyze the optimal dynamics involved in the problem stated.

For concrete calculations we have to specify a functional form for the utility function $U(c_m, k_m, k, \gamma)$ and the scrap value function $S(k, \gamma)$. Let c_a denote consumption of the agricultural good. The utility we have adopted for concrete calculations can be written as:

$$U(c_m, k_m, k, \gamma) := \kappa c_m^\sigma + (1-\kappa)\left((k-k_m)^\beta - \gamma\right)^\sigma \tag{6a}$$

$$S(k, \gamma) := s\left(k^\beta - \gamma\right), \tag{6b}$$

where σ denotes a decreasing marginal utility of the two consumption goods, and κ determines the share of agricultural consumption in the utility function.

Remark 1 This utility function is a simplification of the more general specification such as the constant elasticity of substitution (CES) form. Calculations with the more general CES utility function would however give qualitatively the same results. □

This utility function possesses features that will be useful to elaborate with in the remainder of this paper. For example the exponential functional forms for the instantaneous utility functions imply that a minimal amount of consumption of agricultural good is required for the utility function to be defined at all. Also varying the relative weight κ enables us to discuss different cases that fit better with relatively poor (low κ) or relatively rich countries (high κ).

For solving the optimal control problem (5) we apply Pontryagin's Maximum Principle. Therefore we define the corresponding Hamiltonian and Lagrangian[12]

[12]We formulate the Lagrangian only for the constraint (5f) since it can be shown that the other constraints cannot become active along an optimal solution.

$$\mathcal{H}(k, c_m, k_m, \lambda, \lambda_0) = \lambda_0 \mathcal{U} + \lambda(A_m k_m^\alpha - \delta k - c_m) \tag{7a}$$

$$\mathcal{L}(k, c_m, k_m, \lambda, \mu, \lambda_0) = \mathcal{H}(k, c_m, k_m, \lambda, \lambda_0) + \mu(\theta k_m^\alpha - c_m). \tag{7b}$$

Since the problem is normal (see Appendix B) we can set $\lambda_0 = 1$ and omit it subsequently. Then the necessary optimality conditions are given by the Hamiltonian maximizing condition

$$\begin{pmatrix} c_m^\circ(t) \\ k_m^\circ(t) \end{pmatrix} = \max_{c_m, k_m} \mathcal{H}(k(t), c_m, k_m, \lambda(t)) \tag{8}$$

$$A_m k_m^\alpha - c_m \geq 0,$$

the canonical system

$$\dot{k}(t) = \theta k_m^\alpha(k_m^\circ(t), \gamma) - \delta k(t) - c_m^\circ(t) \tag{9a}$$

$$\dot{\lambda}(t) = \rho\lambda - \frac{\partial \mathcal{H}}{\partial k}(k(t), c_m^\circ(t), k_m^\circ(t), \lambda(t)) \tag{9b}$$

the end time condition

$$\mathcal{H}(k(T), c_m^\circ(T), k_m^\circ(T), \lambda(T)) = \rho \mathcal{S}(k(T)) \tag{10a}$$

and the transversality condition

$$\lambda(T) = \frac{\partial \mathcal{S}}{\partial k}. \tag{10b}$$

In Appendix B these necessary optimality conditions are used to derive the properties of an optimal solution. In the next section the derived results are formulated as specific propositions.

4 Analytical results

This section provides some analytical results that can be derived from problem represented in (5). The proofs are provided in appendix.

In Prop. A.1 we prove that, for a positive minimum subsistence level γ, there exists a non-empty region where the capital dynamics are always negative independent of the chosen value of consumption $c_m(\cdot)$ and manufactured capital $k_m(\cdot)$. As an immediate result this yields the following corollary.

Corollary 1 (Existence of a poverty trap) *In an economy with two economic sectors and a minimum subsistence constraint like the social planner's problem represented in (5) with associated constraints, for positive values of γ, there will exist poverty traps.* □

If the scrap value function is zero we can give a full characterization of the optimal solution for problem (5).

Proposition 1 (Optimal vector field) *Let us consider problem (5) with* $s = 0$*, then for some* $\gamma_0 > 0$ *an optimal solution* $(k^*(\cdot), c_m^*(\cdot), k_m^*(\cdot), T^*)$ *can be characterized as follows. For* $\gamma = 0$ *the optimal solution is uniquely given with* $T^* = \infty$ *and for any* $k_0 > 0$*, with* $k(0) = k_0$

$$\lim_{t\to\infty}(k^*(t), c_m^*(t), k_m^*(t) = (\hat{k}, \hat{c}_m, \hat{k}_m),$$

where $(\hat{k}, \hat{c}_m, \hat{k}_m)$ *is the unique equilibrium of the canonical system (B.7). For* $\gamma < \gamma_0$ *there exists a point* $k_I > \bar{k}$*, such that for* $k(0) = k_I$ *there exist two optimal solutions* $(k_i^*(\cdot), c_{mi}^*(\cdot), k_{mi}^*(\cdot), T_i^*), i = 1, 2,$ *with* $T_1^* = \infty$ *and* $T_2^* < \infty$ *and*

$$\lim_{t\to\infty}(k_1^*(t), c_{m1}^*(t), k_{m1}^*(t) = (\hat{k}, \hat{c}_m, \hat{k}_m)$$
$$k_2^*(T_2^*) = \gamma^{\frac{1}{\beta}},\ c_{m2}^*(T_2^*) = k_{m2}^*(T_2^*) = 0.$$

For $\gamma = \gamma_0$ *the equilibrium state* $\hat{k}_m$ *is a threshold point. The behavior of the unique optimal solution* $(k^*(\cdot), c_m^*(\cdot), k_m^*(\cdot), T^*)$ *is then given by*

$$\lim_{t\to\infty}(k^*(t), c_m^*(t), k_m^*(t)) = (\hat{k}, \hat{k}_m, \hat{c}_m),\ T^* = \infty \qquad k(0) \ge \hat{k}_m$$
$$k^*(T^*) = \gamma^{\frac{1}{\beta}},\ c_m^*(T^*) = k_m^*(T^*) = 0,\ T^* < \infty \qquad \gamma^{\frac{1}{\beta}} \le k(0) < \hat{k}_m.$$

For $\gamma > \gamma_0$ *the optimal solution* $(k^*(\cdot), c_m^*(\cdot), k_m^*(\cdot), T^*)$ *is uniquely given as the finite time solution.*

The bifurcation parameter γ_0 *is characterized as the case when the (unique) equilibrium of the canonical system (B.7) exists and the corresponding unstable path* $(k_u(\cdot), \lambda_u(\cdot))$ *intersects the* k*-axis at the point* $\gamma^{\frac{1}{\beta}}$. □

The proof can be found in Appendix C.

This proposition thus makes the point that the optimal solution to problem (5) will depend on both the actual level of subsistence γ as well as the initial resource stock k_0. In Fig. 2 the main results of Prop. 1 are depicted in a bifurcation diagram for different levels of subsistence γ. Here, the dashed curve lying in region (III) denotes the equilibrium solutions of the optimal system (depending on the other parameter values this equilibrium solution may not be optimal for larger values of γ). The solid black curve marks the states, left of which it is not possible to escape the poverty trap (region (II)). Left of the dotted curve (region (III)) the minimum subsistence constraint is not fulfilled. Hence, as can be seen from this diagram higher levels of subsistence γ effectively increases the region of non-admissibility (I) and the poverty trap region (II).

Fig. 3 shows that the existence of an equilibrium of the canonical system does not imply that the long-run equilibrium solution is superior to the finite time solution. The crucial criterion is the global behavior of the unstable path, as it is formulated in Prop. 1.

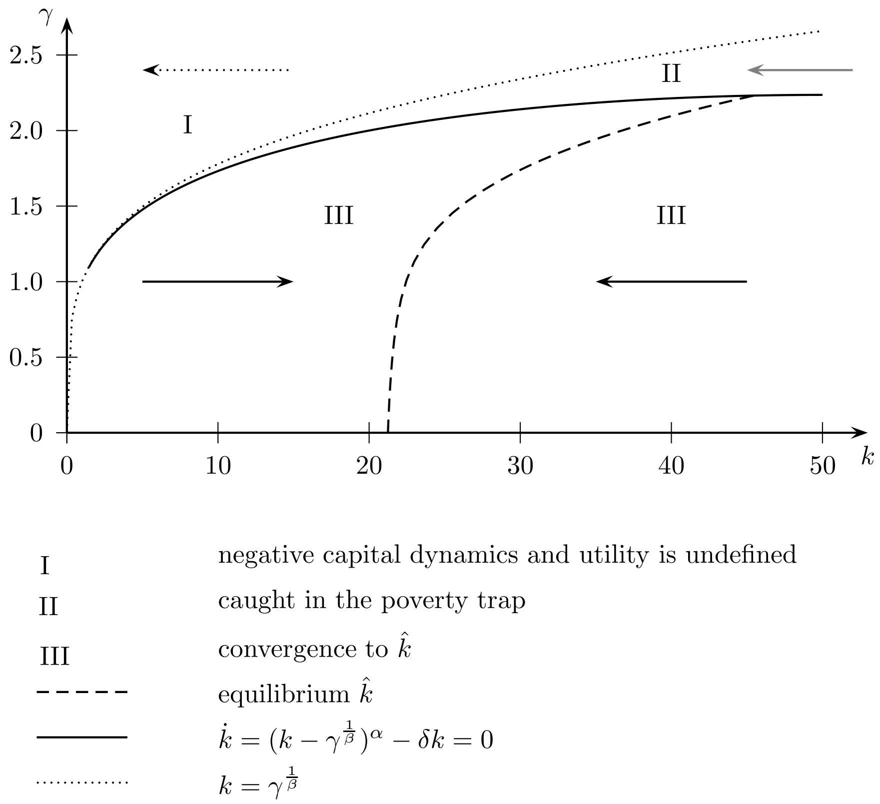
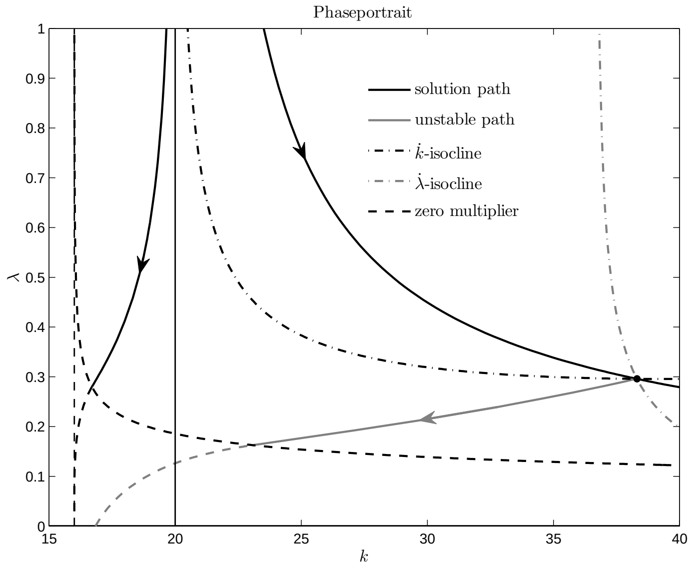
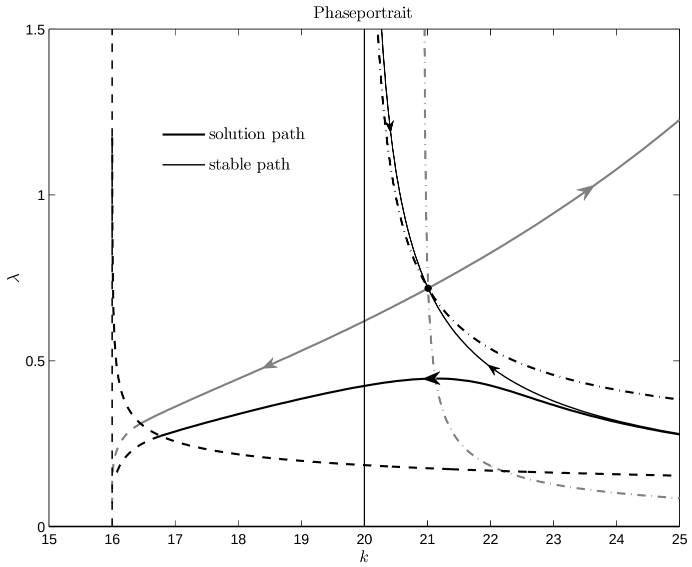


Figure 2: In this figure the different regions of dynamic behavior, depending on the initial state k_0 and actual value of γ, is depicted. The other parameter values are taken from Tab. 1.



(a)



(b)

Figure 3: This figure depicts the cases for $\gamma = 2$, (a) $\rho = 0.01$ and (b) $\rho = 0.125$. The other parameter values are taken from Tab. 1. For both cases an equilibrium exists but whereas in panel (a) the unstable path (gray curve) intersects the k-axis at a point larger than $\gamma^{\frac{1}{\beta}}$ this is not the case in panel (b). Using Prop. 1 this implies the existence of an indifference point in (a) and the superiority of the finite time horizon solution for (b), the finite time horizon solution follows the lower black path leading to $\gamma^{\frac{1}{\beta}}$.

Since small changes of the scrap value function do not change the topological structure of the canonical system we can immediately draw following corollary from Prop. 1.

Corollary 2 (Existence of an indifference point) *In an economy with two economic sectors and a minimum subsistence constraint like the social planner's problem represented in (5) with associated constraints, for small positive values of subsistence parameter γ and the scrap value parameter s, the long run optimal solution will depend on initial conditions. There exists an indifference point k_I for all initial capital stocks $k(0)$, such that when $k(0) > k_I$ there exist an optimal interior infinite time horizon and when $k(0) < k_I$ the optimal solution is to stop within a finite time at the minimum feasible level of capital stock.* □

5 Numerical results

For the numerical calculations we used a method described in Grass (2010). Therefore we have chosen a basic set of parameter values stated in Table 1.[13] For the base case the subsistence level is set to zero yielding a unique long run optimal solution. The phaseportrait of this solution is provided in Fig. B.7 of Appendix. In the subsequent sections we analyze the influence of a change of some of these parameter values and give a consistent interpretation.

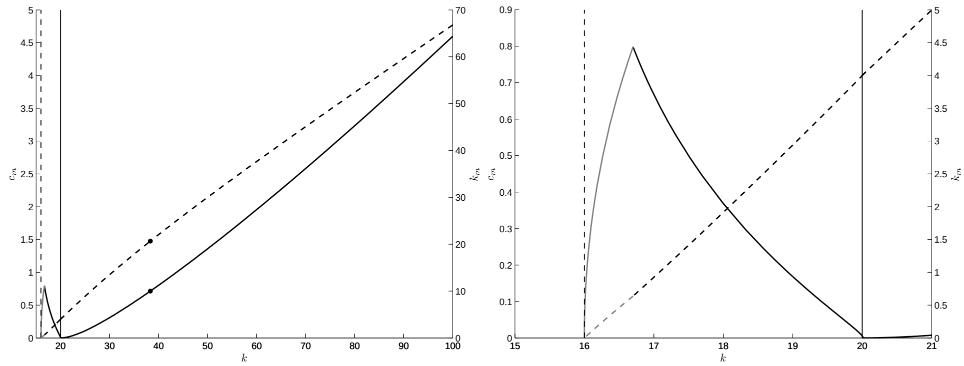
Parameter	Value	Description
ρ	0.01	discount rate
α	0.5	marginal productivity of capital in manuf. sector
β	0.25	marginal productivity of capital in agric. sector
δ	0.1	depriciation rate of capital
γ	0	minimum subsistence level
κ	0.5	share of agric. contribution to overall utility
s	0	parameter of the scrap value function
σ	0.1	diminishing marginal utility of consumption
A_m	1	manuf. productivity parameter
A_a	1	agric. productivity parameter

Table 1: The parameter values for the base case.

5.1 Emergence of poverty traps

Starting from the base case we increase the minimum subsistence level γ up to a value of two, which by Cor. 1 induces the occurrence of a poverty trap. Using the result of Cor. 2 we were able to show the existence of an indifference point

[13] Due to a singularity of the adjoint dynamics for $k = \gamma^{\frac{1}{\beta}}$ the calculations for a positive γ and $s = 0$ have been approximated by some small positive value of s, which by using Cor. 2 does not change the qualitative results.



(a) (b)

Figure 4: In (a) the global behavior of the optimal solution is depicted. Panel (b) shows the neighborhood of the region where poverty traps exist. The point, where the optimal consumption path is not differentiable marks the change from a regime with positive investment to that of zero investment. Due to this change the dynamics of the consumption path switches from some positive to a negative value yielding this non differentiability.

k_I, where the social planner is indifferent between choosing an infinite time horizon or finite time horizon, respectively. See Fig. 4 for the global picture and Fig. 4b for the behavior of the optimal solution in the neighborhood of the region of poverty traps. Due to numerical difficulties it is not possible to calculate the explicit value for the indifference point. But in Fig. 3a it is shown that the unstable path crosses the k axis at a value larger than $\gamma^{\frac{1}{\beta}}$, satisfying therefore the condition for its existence formulated in Prop. 1.

In Fig. 4a the optimal paths are plotted in the state-control space. The solid vertical line represents the initial points left of which one is caught in a poverty trap, i.e. there does not exist any path leading to a long run interior equilibrium. In this area the capital dynamics are always negative ($\dot{k} < 0$) for all admissible controls implying that, in the terminology of Azariadis and Stachurski (2005); there exists a historical self-reinforcing process that deepens poverty. For initial conditions $k(0) \in [0, 16)$, no admissible solutions exist, for $k(0) \in [16, k_I)$ only a solution with a finite time horizon is optimal and for $k(0) > k_I$ the optimal solution has an infinite time horizon. Although it is not possible to calculate an explicit value for the indifference point, by approximation, it can be shown to lie slightly to the right of the solid line, which can be seen by careful examination of Fig. 4b. Thus for any initial state with $16 \leq k(0) \leq k_I$, in our specific example, i.e. between the dashed vertical line and the indifference point there exists a finite optimal time horizon $T^* < \infty$ solving problem (5).

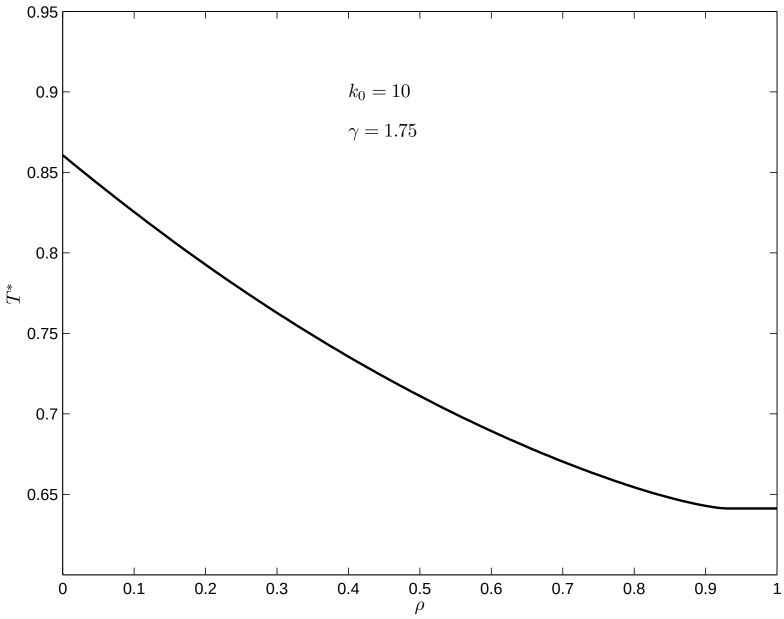


Figure 5: In this figure the optimal finite time horizon is depicted as a function of the discount rate for the base case values but with a subsistence level γ of 1.75 and an inital capital stock in the proximity of the indifference point k_I.

5.2 The discount rate and the optimal time horizon

As argued by (Koopmans, 1973) when imposing subsistence consumption in an exhaustible resource model: "*...discounting of future utilities favors an earlier generation over any surviving later generation and shortens the period of survival. These effects are stronger the higher the discount rate.*". The findings of Koopmans suggests that the choice of the discount rate and the optimal time horizon are intimatly linked. In our model this implies that the for a given initial state the optimal time horizon can be expressed as a function of the discount rate. This behaviour is depicted in Fig. 5. As can be seen from the figure an increase in the discount rate will lead to a decrease in the optimal time horizon.

5.3 Influence of the scrap value function

The scrap value function is a common feature in many models having finite time horizons. In our model the scrap value function could be given several, equally plausible interpretations, as for example the value of switching to an alternative technology or perhaps migrating to a new region. Clearly, setting the scrap value to zero as done in (Koopmans, 1973; Vousden, 1973) imposes a doomsday

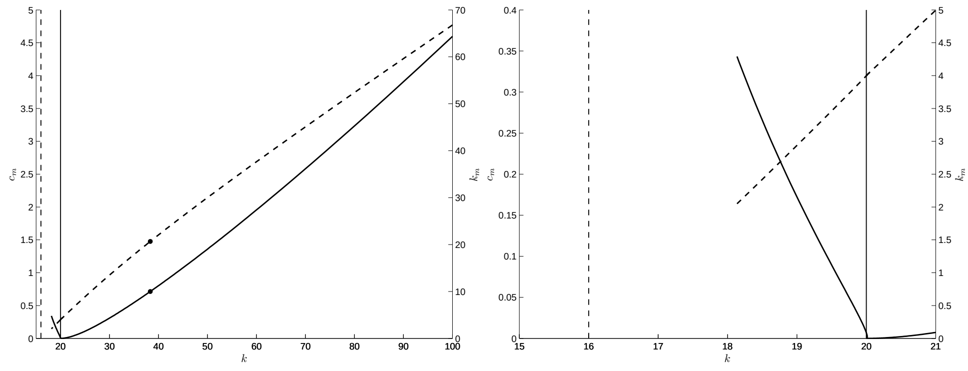


Figure 6: This figure depicts the case of an increased value $s = 15$, yielding a shortened optimal time horizon compared with the case of $s = 0$. Panel (a) covers the state space together with the equilibrium, whereas (b) is zoomed in on the region of the poverty trap.

like scenario.

On the contrary a large value for the scrap value function has an opposite effect. This is due to the fact that the larger the scrap value, the more compelling it becomes, as an alternative to the orginal technology. Increasing, for example the parameter s of the scrap value function (6b), reduces the optimal time horizon in the region of poverty traps as can be seen in Fig. 6.

5.4 Other parameters

We also tested varying several other model parameters. This analysis did however not result in any qualitative changes that were of interest for this paper. For example, the consequences of changing the share of agricultural contribution to overall utility κ had an effect on transitional dynamics as well as the long-run equilibrium solution. Here, a lower value of κ postponed consumption of the manufacturing good which also increased the steady state level of capital.

6 Concluding remarks and possible extensions

We have illustrated how the existence of a minimum subsistence requirement in an economy with two production sectors puts boundaries on the available opportunities for a social planner. This feature also leads to the emergence of a poverty trap like feature within the model.

The concept of a poverty trap as we have defined it in this paper is closely related to problems of inter temporal resource allocation. Common practice in economic literature has been to introduce a discount rate that describes how the utility of a valued commodity declines if the commodity is delivered at some date in the future as opposed to the present. Such a discount rate not only describes how people value present day consumption against future consumption, but also how the present generation discriminates against future generations. Other factors influencing the discount rate are whether we expect the future to be richer or poorer and what the risks of unforeseen events such as extinction possibilities might be. Furthermore there are technical issues associated with the choice of a discount rate; Koopmans (1965) showed that the use of a zero discount rate could imply that no best policy exists because no matter how high the rate of saving is, saving a bit more would always be better.

In addition, problems of inter temporal resource allocation must also consider the planning horizon. Common practice has been to use either an infinite time horizon or a finite time horizon with certain end conditions imposed. Generally, one could argue that an infinite time horizon would be an appropriate approximation of a true, distant and unknown finite time horizon. Further, this is also a convenient approximation due to computational reasons. However as we have shown in this paper, when introducing a minimum consumption requirement, the choice of a discount rate can come to act as a restriction on the planning horizon. The choice of a discount rate will actually have a direct impact on the optimal planning horizon, such that the planning horizon could be described as a function of the discount rate. As discussed in Sect. 2.2, the existence of a minimum subsistence requirement could imply that we become better of by choosing a finite time horizon even when an infinite horizon is possible. It is thus not only quantity but also quality of life that matters.

What ethical considerations should be considered when a morally responsible population decides on the value of future well-being? For example in a rich country it might be ethically indefensible to discount at a high positive rate but in a poor country where resources are scarce, even a low discount rate could imply a saving rate that is unjustifiably high. One way of interpreting the choice of a finite time horizon could be that the present generation is no longer willing to sacrifice present day consumption for the benefit of future generations that might come into existence after the end time. This would be equivalent to setting an infinitely high discount rate after that time. This action might not be ethically defensible for a rich society, but could describe the actual preferences we have towards the future if we were living on the brink of subsistence.

By determining an optimal time horizon we are directly taking into account how the current resources available affect the planning horizon. In doing so we go beyond just setting a discount rate, and instead make inter temporal preferences also contingent upon our current and expected resource state. So how do the time preferences of a poor population compare to that of a rich population where resources are accumulating over time? A poor population trying to survive day-by-day might be unable or even unwilling to sacrifice their scarce amount of resources for the sake of future generations. Due to the risk and uncertainties involving the future, there might be several reasons why this could be logical and perhaps not even questionable on ethical grounds. Can one blame a starving man for not saving enough? The rich population on the other hand does not have the possibility to justify on similar grounds, why they should not sacrifice, at least some arbitrary small amount of their accumulating resources on behalf of future generations. At least, the possibility is open to them.

Several extensions of this research are possible. For example, climate change that hits both sectors differently will affect the boundaries of the problem and the size of the poverty trap. Initial explorations show that, climate change induced damage might increase the region where the problem studied has no solution and decrease the region where it has an infinite horizon solution. The implications of this could be that for the situations when the economy is either very rich or very poor, introducing climate change into the model would only alter the solution quantitatively but not qualitatively. On the contrary, for some intermediate economies, introducing climate change induced damages could imply that an economy that initially was heading towards an infinite horizon interior equilibrium instead might be pushed into a climate induced poverty trap.

Appendix A The maximum capital dynamics

For an analysis of the budget constraint (4) satisfying the subsistence food requirement (3) it is important to consider the maximum capital dynamics defined as

$$K^\circ(k,\gamma) := \max_{c_m,k_m} A_m k_m^\alpha - \delta k - c_m. \tag{A.1}$$

Because (A.1) allows us to derive some principal dynamic features also for the optimally controlled system, where we often make use of the following proposition

Proposition A.1 *Let $0 < \alpha, \beta < 1$ and $\delta > 0$ then the maximum capital dynamics (A.1) is given by*

$$K^\circ(k,\gamma) = A_m(k - \gamma^{\frac{1}{\beta}})^\alpha - \delta k. \tag{A.2}$$

There exists a unique differentiable function $\gamma(k)$ satisfying

$$K^\circ(\gamma(k),\gamma) = 0,$$

which exhibits a unique maximum at

$$\bar{\gamma} = \left(\frac{A_m}{\delta}\left(\left(\frac{\delta}{A_m\alpha}\right)^{\frac{\alpha}{\alpha-1}}\right) - \left(\frac{\delta}{A_m\alpha}\right)^{\frac{1}{\alpha-1}}\right)^\beta$$

for

$$\bar{k} = \frac{A_m}{\delta}\left(\left(\frac{\delta}{A_m\alpha}\right)^{\frac{\alpha}{\alpha-1}}\right)$$

and

$$\gamma(k) = 0 \quad \textit{iff} \quad k = 0 \quad \textit{or} \quad k_1 = \left(\frac{\delta}{A_m}\right)^{\frac{1}{\alpha-1}}.$$

In the region

$$R^+ = \{(k,\gamma):\ 0 < k < \tilde{k},\ 0 \le \gamma < \gamma(k)\}$$

the maximum capital dynamics (A.1) is positive, for every other pair (k,γ) in the positive quadrant and not lying on $\gamma(k)$ (A.1) is negative. □

Corollary 3 *For $0 \le \gamma < \bar{\gamma}$ there exist exactly two values $k_1(\gamma)$ and $k_2(\gamma)$ satisfying*

$$0 \le k_1(\gamma) < k_2(\gamma) \le \left(\frac{\delta}{A_m}\right)^{\frac{1}{\alpha-1}}$$

with

$$K^\circ(\gamma(k),\gamma) = \begin{cases} >0 & k_1(\gamma) < k < k_2(\gamma) \\ =0 & k = k_i(\gamma),\ i=1,2 \\ <0 & 0 < k < k_1(\gamma) \quad or \quad k > k_2(\gamma). \end{cases} \qquad \square$$

Proof To derive the (A.2) we note that the maximum of the dynamics is achieved, when c_m is at its minimum, i.e., $c_m = 0$, and k_m receive its maximum, which is constrained by the subsistence inequality (3) and yielding

$$k_m = k - \gamma^{\frac{1}{\beta}}.$$

To prove the existence of a function $\gamma(k)$ we apply the implicit function theorem to the equation

$$K^\circ(k,\gamma) = 0. \tag{A.3}$$

Therefore we calculate the partial derivatives

$$\frac{\partial K^\circ}{\partial k}(k,\gamma) = A_m\alpha(k-\gamma^{\frac{1}{\beta}})^{\alpha-1} - \delta \tag{A.4}$$

$$\frac{\partial K^\circ}{\partial \gamma}(k,\gamma) = -\frac{A_m\alpha}{\beta}(k-\gamma^{\frac{1}{\beta}})^{\alpha-1}\gamma^{\frac{1-\beta}{\beta}} < 0, \tag{A.5}$$

proving that

$$\left(\frac{\partial K^\circ}{\partial k}, \frac{\partial K^\circ}{\partial \gamma}\right) \neq (0,0).$$

Therefore and since for $\gamma = 0$ (A.3) exhibits the solution $k = 0$ the implicit function theorem assures the existence of a continuously differentiable curve $\gamma(k)$ for all $k \geq 0$ satisfying

$$K^\circ(k,\gamma(k)) = 0.$$

The derivative of $\gamma(k)$ is given as

$$\gamma'(k) = -\frac{\frac{\partial K^\circ}{\partial k}(k,\gamma)}{\frac{\partial K^\circ}{\partial \gamma}(k,\gamma)} = \frac{A_m\alpha(k-\gamma^{\frac{1}{\beta}})^{\alpha-1} - \delta}{\frac{A_m\alpha}{\beta}(k-\gamma^{\frac{1}{\beta}})^{\alpha-1}\gamma^{\frac{1-\beta}{\beta}}}. \tag{A.6}$$

For (A.6) to become zero the two equations have to be satisfied

$$A_m(k-\gamma^{\frac{1}{\beta}})^{\alpha} - \delta k = 0$$
$$A_m\alpha(k-\gamma^{\frac{1}{\beta}})^{\alpha-1} - \delta = 0$$

yielding the unique solution

$$\bar{\gamma} = \left(\frac{A_m}{\delta}\left(\left(\frac{\delta}{A_m\alpha}\right)^{\frac{\alpha}{\alpha-1}}\right) - \left(\frac{\delta}{A_m\alpha}\right)^{\frac{1}{\alpha-1}}\right)^{\beta}$$

$$\bar{k} = \frac{A_m}{\delta}\left(\left(\frac{\delta}{A_m\alpha}\right)^{\frac{\alpha}{\alpha-1}}\right).$$

To show that $\gamma(\bar{k})$ is global maximum we calculate the second derivative, yielding

$$\gamma''(\bar{k}) = -\frac{\frac{\partial^2 K^\circ}{\partial k^2}(k,\gamma)}{\frac{\partial K^\circ}{\partial \gamma}(k,\gamma)} - \gamma'(\bar{k})\frac{\frac{\partial^2 K^\circ}{\partial\gamma\partial k}(k,\gamma) + \frac{\partial^2 K^\circ}{\partial \gamma^2}(k,\gamma)\gamma'(\bar{k})}{\frac{\partial K^\circ}{\partial \gamma}(k,\gamma)}$$

$$= -\frac{\frac{\partial^2 K^\circ}{\partial k^2}(k,\gamma)}{\frac{\partial K^\circ}{\partial \gamma}(k,\gamma)} < 0.$$

Since $\gamma(k) = 0$ exhibits a second solution

$$\tilde{k} = \left(\frac{\delta}{A_m}\right)^{\frac{1}{\alpha-1}},$$

this ends the proof of the first part of Prop. A.1

Noting that

$$K^\circ(k,0) = A_m k^\alpha - \delta k > 0,\ k \in (0,\tilde{k}),$$

together with the previously derived shape of the function $\gamma(k)$, proves the second part of the proposition. ■

Remark 2 An analogous result can be proved for the model, where labor is explicitly included, which exhibits the structural similarity to the simplified model (Poof available from authors). □

Appendix B Necessary optimality conditions

In the subsequent sections we derive the optimal control values and the corresponding representations of the canonical system for all possible combinations of active/inactive constraints.

Abnormal case

The abnormal case $\lambda_0 = 0$ can be excluded since otherwise the adjoint dynamics reduces to

$$\dot{\lambda}(t) = (\rho + \delta)\lambda(t).$$

But then the limiting transversality condition

$$\lim_{t\to\infty} \mathrm{e}^{-\rho t}\,\lambda(t) = 0$$

is only satisfied for $\lambda(0) = 0$ which yields $\lambda(t) = 0$ for all $t \geq 0$ and this violates $(\lambda_0, \lambda(t)) \neq (0,0)$.

Hessian Matrix

To prove the (local) optimality of the control values the second order derivative of the Lagrangian $\mathcal{L}$ has to be considered. Using the functional form (6a) of the utility yields

$$\frac{\partial^2 \mathcal{L}}{\partial k^2} = \frac{\partial^2 U}{\partial k^2} < 0 \tag{B.1}$$

$$\frac{\partial^2 \mathcal{L}}{\partial c_m^2} = \frac{\partial^2 U}{\partial c_m} < 0 \tag{B.2}$$

$$\frac{\partial^2 \mathcal{L}}{\partial k_m^2} = \frac{\partial^2 U}{\partial k_m^2} + (\lambda + \mu) A_m \alpha(\alpha - 1) k_m^{\alpha-2} < 0 \tag{B.3}$$

$$\frac{\partial^2 \mathcal{L}}{\partial k \partial k_m} = \frac{\partial^2 U}{\partial k \partial k_m} > 0 \tag{B.4}$$

$$\frac{\partial^2 \mathcal{L}}{\partial k \partial c_m} = \frac{\partial^2 \mathcal{L}}{\partial k_m \partial c_m} = 0, \tag{B.5}$$

yielding a (strictly) negative definite Hessian matrix

$$L = \begin{pmatrix} \frac{\partial^2 \mathcal{L}}{\partial k^2} & 0 & \frac{\partial^2 \mathcal{L}}{\partial k \partial k_m} \\ 0 & \frac{\partial^2 \mathcal{L}}{\partial c_m^2} & 0 \\ \frac{\partial^2 \mathcal{L}}{\partial k \partial k_m} & 0 & \frac{\partial^2 \mathcal{L}}{\partial k_m^2} \end{pmatrix},$$

since

$$\det L^{(1)} = \frac{\partial^2 \mathcal{L}}{\partial k^2} < 0$$

$$\det L^{(2)} = \frac{\partial^2 \mathcal{L}}{\partial k^2} \frac{\partial^2 \mathcal{L}}{\partial c_m^2} > 0$$

$$\det L^{(3)} = \frac{\partial^2 \mathcal{L}}{\partial k^2} \frac{\partial^2 \mathcal{L}}{\partial c_m^2} A_m (\lambda + \mu) A_m \alpha(\alpha - 1) k_m^{\alpha-2} \leq 0,$$

with

$$L^{(i)}, \quad i = 1, 2, 3$$

the ith leading principal submatrix.

Case $0 < c_m < A_m k_m^\alpha$ and $p_a > \gamma$

Using the functional form (6a) of the the utility function we find

$$\frac{\partial \mathcal{H}}{\partial c_m} = \sigma\kappa c_m^{\sigma-1} - \lambda = 0 \tag{B.6a}$$

$$\frac{\partial \mathcal{H}}{\partial k_m} = -\sigma(1-\kappa)\beta k_a^{\beta-1}\left(k_a^\beta - \gamma\right)^{\sigma-1} + \lambda A_m \alpha k_m^{\alpha-1} = 0 \tag{B.6b}$$

yielding an explicit solution for (B.6a)

$$c_m^\circ = \left(\frac{\lambda}{\sigma\kappa}\right)^{\frac{1}{\sigma-1}}. \tag{B.6c}$$

Since (B.6b) cannot be solved explicitely the corresponding canonical system can be stated as a system of differential algebraic equations (DAEs)

$$\dot{k} = A_m k_m^\alpha - \delta k - c_m \tag{B.7a}$$

$$\dot{\lambda} = (\rho+\delta)\lambda - \sigma(1-\kappa)\beta k_a^{\beta-1}\left(k_a^\beta - \gamma\right)^{\sigma-1} \tag{B.7b}$$

$$0 = -\sigma(1-\kappa)\beta k_a^{\beta-1}\left(k_a^\beta - \gamma\right)^{\sigma-1} + \lambda A_m \alpha k_m^{\alpha-1} \tag{B.7c}$$

Case: $c_m = A_m k_m^\alpha$ and $(k-k_m)^\beta > \gamma$

The corresponding equations become

$$\frac{\partial \mathcal{L}}{\partial c_m} = \sigma\kappa c_m^{\sigma-1} - \lambda - \mu = 0 \tag{B.8a}$$

$$\frac{\partial \mathcal{L}}{\partial k_m} = -\sigma(1-\kappa)\beta k_a^{\beta-1}\left(k_a^\beta - \gamma\right)^{\sigma-1} + (\lambda+\mu) A_m \alpha k_m^{\alpha-1} = 0 \tag{B.8b}$$

$$A_m k_m^\alpha - c_m = 0. \tag{B.8c}$$

Substituting (B.8a) and (B.8c) into (B.8b) yields an equation in k_m

$$0 = -\sigma(1-\kappa)\beta k_a^{\beta-1}\left(k_a^\beta - \gamma\right)^{\sigma-1} + \sigma\kappa (k_m^\alpha)^{\sigma-1} A_m \alpha k_m^{\alpha-1}. \tag{B.8d}$$

From (B.8d) the optimal control value of k_m is only implicitly given. Thus the canonical system (9) is given as the following DAE

$$\dot{k} = -\delta k \tag{B.9a}$$

$$\dot{\lambda} = (\rho+\delta)\lambda - \sigma(1-\kappa)\beta k_a^{\beta-1}\left(k_a^\beta - \gamma\right)^{\sigma-1} \tag{B.9b}$$

$$0 = -\sigma(1-\kappa)\beta k_a^{\beta-1}\left(k_a^\beta - \gamma\right)^{\sigma-1} + \sigma\kappa (k_m^\alpha)^{\sigma-1} A_m \alpha (k_m)^{\alpha-1}. \tag{B.9c}$$

Phase portrait of the canonical system

Proposition B.2 (Interior control values) *For $\gamma < \bar{\gamma}$ the canonical system (B.7) exhibits a (unique) saddle $(\hat{k}, \hat{\lambda})$ with corresponding control values $(\hat{c}_m, \hat{k}_m)$ given by (B.6c) and (B.7c) and a stable manifold $\mathrm{W}^s(\hat{c}_m, \hat{k}_m)$ and an unstable manifold $\mathrm{W}^u(\hat{c}_m, \hat{k}_m)$. The $\dot{k}$-isocline $i^{(k)}$ and $\dot{\lambda}$-isocline $i^{(\lambda)}$ satisfy the following conditions*

$$\lim_{k \downarrow \bar{k}_1(\gamma)} i^{(k)}(k) = \lim_{k \uparrow \bar{k}_2(\gamma)} i^{(k)}(k) = \infty$$

$$\lim_{k \downarrow \hat{k}_m + \gamma^{\frac{1}{\beta}}} i^{(\lambda)}(k) = \infty$$

$$\lim_{k \to \infty} i^{(\lambda)}(k) = 0$$

with

$$\bar{k}_1(\gamma) \leq \hat{k}_m + \gamma^{\frac{1}{\beta}} \leq \hat{k} \leq \bar{k}_2(\gamma)$$

Let $\lambda^s(k)$ and $\lambda^u(k)$ be the functional representation of the stable and unstable manifold, respectively then these functions satisfy

$$\lim_{k \downarrow \bar{k}_1(\gamma)} \lambda^s(k) = \infty$$

$$\lim_{k \to \infty} \lambda^s(k) = 0$$

$$\lim_{k \to \check{k}} \lambda^u(k) = 0 \quad \textit{with} \quad \gamma^{\frac{1}{\beta}} < \check{k} < \hat{k}. \qquad \square$$

Proof At this place we only give a brief sketch of the full proof which is available from the authors. Setting the canonical system (B.7) to zero yields that k_m is constant

$$k_m = \left(\frac{\rho + \delta}{A_m \alpha}\right)^{\frac{1}{\alpha - 1}}. \tag{B.10a}$$

and that the following equation has to be satisfied

$$F(k, \gamma) := A_m \left(\frac{\rho + \delta}{A_m \alpha}\right)^{\frac{\alpha}{\alpha - 1}} - \delta k - \Omega (k - k_m)^{\frac{\beta - 1}{\sigma - 1}} \left((k - k_m)^\beta - \gamma\right) \tag{B.10b}$$

with

$$\Omega := \left(\frac{(1 - \kappa)\beta}{\kappa(\rho + \delta)}\right)^{\frac{1}{\sigma - 1}}.$$

To prove the existence of a zero for (A.3) and $\gamma = 0$ we find

$$\begin{aligned}
\lim_{k \to k_m} F(k, 0) &= A_m \left(\frac{\rho + \delta}{A_m \alpha}\right)^{\frac{\alpha}{\alpha - 1}} - \delta \left(\frac{\rho + \delta}{A_m \alpha}\right)^{\frac{1}{\alpha - 1}} \\
&= \left(\frac{\rho + \delta}{A_m \alpha}\right)^{\frac{1}{\alpha - 1}} \left(\frac{\rho + \delta}{\alpha} - \delta\right) > 0.
\end{aligned}$$

From Cor. 3 we derive that for $k > k_2(0)$, $F(k,0) < 0$ holds, which due to the intermediate value theorem proves the existence of a $\hat{k}$ with $F(\hat{k},0) = 0$. Moreover since $F(k,0)$ is strictly decreasing by (B.11a) $\hat{k}$ is the only zero.

Since the matrix of the partial derivatives of (B.10b)

$$\frac{\partial F}{\partial k}(k,\gamma) = -\delta - \Omega\frac{\beta-1}{\sigma-1}(k-k_m)^{\frac{\beta-\sigma}{\sigma-1}}\left((k-k_m)^\beta - \gamma\right) - \Omega(k-k_m)^{\frac{\beta-1}{\sigma-1}}\beta(k-k_m)^{\beta-1} < 0 \tag{B.11a}$$

$$\frac{\partial F}{\partial \gamma}(k,\gamma) = \Omega(k-k_m)^{\frac{\beta-1}{\sigma-1}} > 0 \tag{B.11b}$$

has full rank the implicit function theorem and using (B.11) the existence of a unique equilibrium, given by a differentiable function $\hat{k}(\gamma)$ can be proved for any $0 \le \gamma \le \bar{\gamma}$.

To derive the properties of the isoclines the implicit function theorem is applied to the system of equations

$$E^k(k,k_m,\lambda,c_m) := \begin{pmatrix} A_m P_m(k_m) - \delta k - c_m \\ V(k_a) + \lambda A_m \dfrac{\partial P_m}{\partial k_m} \\ W(c_m) - \lambda \end{pmatrix}$$

and

$$E^\lambda(k,k_m,\lambda) := \begin{pmatrix} (\rho+\delta)\lambda + V(k_a) \\ V(k_a) + \lambda A_m \dfrac{\partial P_m}{\partial k_m} \\ W(c_m) - \lambda \end{pmatrix}$$

with

$$V(k_a) := -\sigma(1-\kappa)\beta k_a^{\beta-1}\left(k_a^\beta - \gamma\right)^{\sigma-1}$$
$$W(c_m) := \sigma\kappa c_m^{\sigma-1}$$

Together with the results of Prop. A.1 the stated properties of the isoclines and (un)stable manifold can then be derived. ■

Proposition B.3 (Boundary control value) *For the canonical system (B.9) the $\dot{k}$-isocline is given by $k = 0$, the $\dot{\lambda}$-isocline $i^{(\lambda)}$ and the zero-multiplier $i^{(\mu)}$ isocline satisfy the following conditions*

$$\lim_{k \downarrow \gamma^{\frac{1}{\beta}}} i^{(\lambda)}(k) = \lim_{k \downarrow \gamma^{\frac{1}{\beta}}} i^{(\mu)}(k) = \infty$$
$$\lim_{k\to\infty} i^{(\lambda)}(k) = \lim_{k\to\infty} i^{(\mu)}(k) = 0$$

There exists a unique intersection point $\check{k} > \hat{k}$ satisfying $i^{(\lambda)}(\check{k}) = i^{(\mu)}(\check{k})$. □

At this place the proof of Prop. B.3 is omitted but is available from the authors.

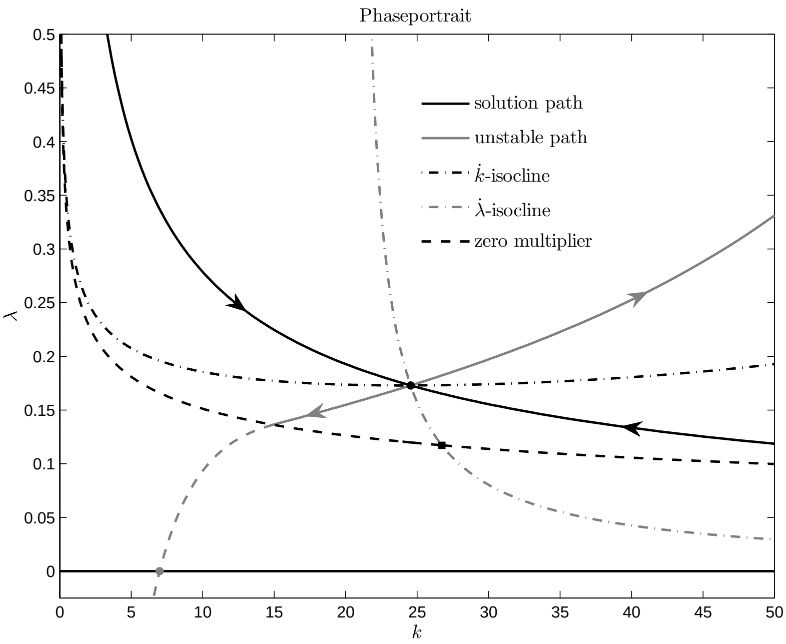


Figure B.7: This figure depicts the paradigmatic phase portrait of the canonical system for $\gamma = 0$, with $\dot{k}$-isocline, $\dot{\lambda}$-isocline interior controls, $\dot{\lambda}$-isocline boundary controls, zero multiplier curve, stable path, unstable path interior controls, unstable path boundary controls. The equilibrium is denoted by the black dot, the intersection of the $\dot{\lambda}$-isoclines and the zero multiplier curve is shown by the square and the gray dot shows intersection point of the unstable path with the k axis.

Appendix C Proof of Prop. 1

For the proof we distinguish the two cases $\gamma = 0$ and $\gamma > 0$.

Case $\gamma = 0$ The necessary optimality conditions for an optimal finite time horizon solution yields

$$\mathcal{H}(T^*) = 0,$$

which for $\gamma = 0$ can only be achieved for

$$k(T^*) = k_m(T^*) = c_m(T^*) = 0.$$

But for $k(0) > 0$ there exists no admissible solution reaching $k = 0$ in finite time, since

$$\dot{k}(t) = A_m k_m(t)^\alpha - \delta k(t) - c_m(t) \geq -\delta k(t).$$

But an admissible solution converging to $k = 0$ does not satisfy the transversality conditions. To see that we use the results of Prop. B.3, where we proved that a path converging to $k = 0$ crosses the zero multiplier curve and therefor the mixed control constraint becomes active. Moreover we see that in this region the adjoint dynamics (B.9b) is strictly negative and moreover from the adjoint equation we then find for some (negative) constant c

$$\lim_{t\to\infty} \lambda(t) \leq c\, \mathrm{e}^{(\rho+\delta)t}$$

and therefore

$$\lim_{t\to\infty} \mathrm{e}^{-\rho t}\, \lambda(t) \leq c\, \mathrm{e}^{\delta t} = -\infty.$$

Case $0 < \gamma < \gamma_0$ For a small positive γ the dynamic structure of the canonical system shows that the (interior) equilibrium still persists with a saddle path asymptotic to the orthogonal line at $\bar{k}_1(\gamma)$. Moreover for some finite time T and some $k(0) < \hat{k}$ there exists a path $(k(\cdot), \lambda(\cdot))$ ending up at $k = \gamma^{\frac{1}{\beta}}$. This path can be extended up to some $k > \bar{k}_1(\gamma)$, where it crosses the $\dot{k}$-isocline. At this point the solution converging to the equilibrium exhibits a greater objective value, due to the strict concavity of the Hamiltonian with respect to the costate, which achieves its minimum at the $\dot{k}$-isocline. But at $k = \bar{k}_1(\gamma)$ the objective value of the finite time horizon solution is positive and and since objective value of the saddle path solution converges to zero at this point, the finite time horizon solution is better in a neighborhood of $k = \bar{k}_1(\gamma)$. Due to the continuity of the Hamiltonian there exists a unique indifference point k_I, separating the finite and infinite time horizon solution. The phaseportrait is drawn in Fig. B.7.

Case $\gamma = \gamma_0 > 0$ Considering the phase portrait of the canonical system we see that the path ending at $\gamma^{\frac{1}{\beta}}$ and $\lambda = 0$ is restricted to a finite region as long as the unstable path of the equilibrium crosses the k-axis at some value $k > \gamma^{\frac{1}{\beta}}$. In any other case such a path exists for every initial $k \geq \gamma^{\frac{1}{\beta}}$. Specifically the path exists for $k(0) = \hat{k}$. Now we can argue that due to the strict convexity of the Hamiltonian (see, e.g., Grass et al., 2008) with respect to the costate the finite time horizon solution is better for $k(0) = \hat{k}$ than the infinite time horizon solution. But then this already follows for every $k(0) \geq \gamma^{\frac{1}{\beta}}$. Thus the case $\gamma = \gamma_0 > 0$, where the unstable path of the equilibrium crosses the k-axis exactly at the value $\gamma^{\frac{1}{\beta}}$, denotes the bifurcation value, where for all $\gamma \geq \gamma_0$ the finite time horizon gives the maximum, which ends the proof.

References

Azariadis, C., Stachurski, J., 2005. Poverty traps, in: Aghion, P., Durlauf, S. (Eds.), Handbook of Economic Growth. Elsevier. volume 1 of *Handbook of Economic Growth*. chapter 5, pp. 295–379.

Dasgupta, P., 1978. Fairness between generations and the social rate of discount. Resources Policy 4, 172–177.

Echevarria, C., 1997. Changes in Sectoral Composition Associated with Economic Growth. International Economic Review 38, 431–452.

Galor, O., Weil, D.N., 2000. Population , Technology , and Growth : From Maithusian Stagnation to the Demographic Transition and Beyond. American Economic Review 90, 806–828.

Gollin, D., Parente, S., Rogerson, R., 2002. The Role of Agriculture in Development. The American Economic Review 92, 160–164.

Gollin, D., Parente, S., Rogerson, R., 2007. The food problem and the evolution of international income levels. Journal of Monetary Economics 54, 1230–1255.

Grass, D., 2010. Numerical computation of the optimal vector field in a fishery model. Technical Report. ORCOS, Institute of Mathematical Methods in Economics, Vienna University of Technology.

Grass, D., Caulkins, J., Feichtinger, G., Tragler, G., Behrens, D., 2008. Optimal Control of Nonlinear Processes: With Applications in Drugs, Corruption, and Terror. Springer Verlag, Berlin.

IPCC, 2007. Climate Change 2007: The Fourth Assessment Report of the Intergovernmental Panel on Climate Change. Cambridge University Press, Cambridge.

Irz, X., Roe, T., 2005. Seeds of growth? Agricultural productivity and the transitional dynamics of the Ramsey model. European Review of Agricultural Economics 32, 143–165.

Johnston, B.F., Mellor, J.W., 1961. The Role of Agriculture in Economic Development. American Economic Review 51, 566–593.

King, R.G., Rebelo, S.T., 1993. Transitional Dynamics and Economic Growth in the Neoclassical Model. The American Economic Review 81, 908–931.

Kiseleva, T., Wagener, F., 2010. Bifurcations of optimal vector fields in the shallow lake system. Journal of Economic Dynamics and Control 34, 825–843.

Kongsamut, P., Rebelo, S., Xie, D., 2001. Beyond Balanced Growth. The Review of Economic Studies 68, 869–882.

Koopmans, T.C., 1965. On the concept of optimal economic growth, in: The Econometric Approach to Development Planning. North Holland, Amsterdam, pp. 225–300.

Koopmans, T.C., 1973. Some observations on optimal economic resource extraction. Cowles Foundation Paper.

Koopmans, T.C., 1974. Proof for a Case where Discounting Advances the Doomsday. The Review of Economic Studies 41, 117.

Koopmans, T.C., 1977. Concepts of optimality and their uses. American Economic Review 67, 261–274.

Kraay, a., Raddatz, C., 2007. Poverty traps, aid, and growth. Journal of Development Economics 82, 315–347.

Kumar, R.C., Naqib, F.M., 1993. On the determination of optimal time horizon in a control problem in natural resource economics. International Journal of Mathematics and Mathematical Sciences 18.

Kuznet, S., 1966. Modern economic growth: rate, structure, and spread. Yale University Press, New Haven.

Lewis, A.W., 1955. The Theory of Economic Growth. Routledge, London.

Matsuyama, K., 1992. Agricultural productivity, comparative advantage, and economic growth. Journal of Economic Theory 58, 317–334.

Schultz, T.W., 1953. The Economic Organization of Agriculture. McGraw-Hill, New York.

Steger, T., 2000. Economic growth with subsistence consumption. Journal of Development Economics 62, 343–361.

Timmer, C.P., 2009. A World without Agriculture? The Historical Paradox of Agricultural Development. AEI Development Policy Outlook series. 1. American Enterprise Institute for Public Policy Research.

Timmer, P.C., 1988. The agricultural transformation, in: Chenery, H., Srinivasan, T. (Eds.), Handbook of Development Economics. Elsevier. volume 1 of *Handbook of Development Economics*. chapter 8, pp. 275–331.

Vousden, N., 1973. Basic theoretical issues of resource depletion. Journal of Economic Theory 6, 126–143.

Wagener, F., 2006. Skiba points for small discount rates. Journal of Optimization Theory and Applications 128, 261–277.

Withagen, C., 1981. The optimal exploitation of exhaustible resources, a survey. De Economist 129.

World Bank, 2008. World development report 2008. World Bank, Washington, DC.