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Energy Balance Climate Models and the Spatial Structure of Optimal Mitigation Policies

William Brock[*], Gustav Engstrom[†]and Anastasios Xepapadeas[‡]

January 23, 2012

Abstract

We develop a one-dimemsional energy balance climate model with heat transportation across locations. We introduce the concept of potential world GDP at time t, and we introduce, through the temperature function, spatial characteristics into the damage function which make damages latitude dependent. We solve the social planner's problem and characterize the competitive equilibrium. We define optimal taxes on fossil fuels and profit taxes on firms that extract fossil fuels. Our results suggest that if the implementation of international transfers across latitudes is not possible, then optimal taxes are spatially non homogeneous and tend to be lower at the poor latitudes. The degree of spatial differentiation of optimal taxes depend on heat transportation. We also locate sufficient conditions for optimal mitigation policies to have rapid ramp-up initially and then decrease over time. By employing the properties of the spatial model and approximating solutions, we show how to study the impact of thermal transport across latitudes on welfare inequality.

Keywords: Energy Balance Climate Models, Heat Diffusion, Temperature Distribution, Spatial Optimal Taxes

JEL Classification: Q54, Q58

1 Introduction

The impact of climate change is expected to have a profound regional structure in terms of temperature and damage differentials across geographical

regions.[1] The spatial dimension of damages can be associated with two main factors: (i) Natural mechanisms which produce a spatially *non-uniform* distribution of the surface temperature across the globe. These mechanisms relate mainly to the heat flux that balances incoming and outgoing radiation and in the differences among the local heat absorbing capacity - the local albedo - which is relatively lower in ice covered regions; (ii) economic related forces which determine the damages that a regional (local) economy is expected to suffer from a given increase of the local temperature. These damages depend mainly on the production characteristics (e.g. agriculture vs services) or local natural characteristics (e.g. proximity to the sea and elevation from the sea level). The interactions between the spatially non-uniform temperature distribution and the spatially non uniform economic characteristics will finally shape the spatial distribution of damages.

Existing literature and in particular the DICE/RICE models (e.g. Nordhaus and Boyer 2000, Nordhaus, 2007, 2010, 2011) provide a spatial distribution of damages where the relatively higher damages from climate change are concentrating in the zones around the equator.[2] These models as well as the big majority of Integrated Assessment Models (IAMs) do not account however for the first factor, the natural mechanism generating temperature distribution across the globe. DICE/RICE type models do not include the spatial transportation of heat, or albedo differentials across locations, and perform their analysis in terms of the global mean surface temperature which does not vary across regions during their planning horizons.

In climate science terminology the IAMs with a carbon cycle are zero-dimensional models and they do not include spatial effects due to heat diffusion across space. This can be contrasted to the one- or two-dimensional energy balance climate models (EBCMs) developed by climate scientists which model heat diffusion across latitudes or across latitudes and longitudes (e.g. Budyko 1969, Sellers 1969,1976, North 1975 a,b, North et al. 1981, Kim and North 1992, Wu and North 2007). One-dimensional EBCMs predict a concave temperature distribution across latitudes with the maximum temperature at the equator. This non uniform temperature distribution is important for understanding the so called "temperature anomaly" which is the difference between the temperature distribution at a given benchmark period and the current period. Data indicate (NASA) that since 1880 the anomaly has been higher in high latitude zones, relative to zones around the equator, which suggest spatial non-uniformity in the distribution of temper-

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[1] Detailed reports of climate change effects on different parts of the world can be found at

http://www.metoffice.gov.uk/climate-change/policy-relevant/obs-projections-impacts

[2] For example, Nordhaus's RICE 2010 divides the world into US, EU, Japan, Russia, Eurasia, China, India, Middle East, Africa, Latin America, Other high income, Other developing Asia.

ature.

The temperature anomaly is however the basis for estimating regional damages. Regional damages are obtained by mapping a given change in the temperature of a region relative to a benchmark period - the temperature anomaly - to the damages that this change is expected to bring given the characteristics of the region's economy. In the context of a zero-dimensional model this temperature anomaly will be spatially homogeneous, or flat across regions, since climate change acts on the global average temperature which is spatially homogeneous. In the context of a one- or two-dimensional model climate change acts on the spatially non homogeneous temperature distribution. This is expected to result in a spatially non homogeneous distribution of the temperature anomaly which in turn will differentiate the distribution of damages from those implied by a zero dimensional model.

In this paper we study the economics of climate change by coupling a one-dimensional EBCM with heat diffusion and albedo differentiation across latitudes, with an economic growth model. We believe that this approach that integrates solution methods for one-dimensional spatial climate models, that may be new to economics, with methods of solving economic models, can provide new insights regarding issues such as the spatiotemporal of optimal mitigation policy and the spatial distribution of damages, relative to the more conventional integrated assessment models with carbon cycle but without heat diffusion.

Thus, in the context described above the main contribution of our paper is to couple spatial climate models, with economic models, and use these spatial climate models in order to achieve three objectives.

The first objective is to show how heat transport across latitudes matters regarding the prediction of the spatial distribution and the corresponding temporal evolution of temperature, damages and optimal mitigation efforts. In pursuing this objective we endogenously derive temperature and damage distributions, climate response functions that describe the impact of increasing atmospheric carbon dioxide stock on temperature and damages at a specific latitude, as well as measures of spatial inequalities across latitudes. As our result show heat transport explicitly affects the spatial distribution of temperature and damages, thus its omission by zero-dimensional models introduces a bias. Using the coupled one-dimensional model we derive a well defined distribution of the surface temperature with higher temperature in zones around the equator. Furthermore the dynamic nature of our model allows us to study the temporal evolution of this distribution. In contrast zero-dimensional models provide the global average temperature and its evolution. We also derive the spatial distribution of damages resulting from the interactions between heat transport and local economic characteristics. As far as we know, this is the first time that the spatial distribution of surface temperature and damages, and their temporal evolutions are determined endogenously in the conceptual framework of a coupled EBCM - economic

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growth model. We therefore believe that this aspect is a contribution of our paper relative to the traditional IAM with regional disaggregation but without the natural mechanism of heat transport across locations.

The second objective is to provide insights regarding the optimal spatial and temporal profile for current and future mitigation, when thermal transport across latitudes is taken into account. Regarding the spatial profile of fossil fuel taxes our result suggest higher tax rates for wealthier geographical zones due to practical inability of implementing the international transfers needed to implement a competitive equilibrium associated with the Pareto optimum that is attained when welfare weights are Negishi weights. Our one-dimensional model allows us to show how heat diffusion across geographical zones impacts the size of the spatial differentiation of fossil fuel taxes between poor and wealthy regions. The result that in the absence of international transfers a spatially uniform optimal mitigation is not possible has first noted by Chichilnisky and Heal (1994), our result provides new insights into this issue by characterizing the spatial distribution of fossil fuel taxes and linking the degree of spatial differentiation of optimal fossil fuel taxes to the diffusion of heat.

Regarding the temporal profile of optimal mitigation, It seems that among economists dealing with climate change on the mitigation side the debate has basically settled on whether to increase mitigation efforts that is, carbon taxes, gradually (e.g. Nordhaus 2007, 2010, 2011), or whether we should mitigate rapidly (e.g. Stern 2006, Weitzman 2009 a,b). Carey (2011) quotes Robert Mendelsohn as stating that:

> "The debate is how much and when to start. If you believe that there are large damages, you would want more drastic immediate action. The Nordhaus camp, however, says we would start modestly and get tougher over time".

In this paper we locate sufficient conditions for profit taxes on fossil fuel firms to be decreasing over time and for unit taxes on fossil fuels to grow over time less than the rate of return on capital. We also locate sufficient conditions for the tax schedule to be increasing according to the gradualist approach.

The third objective is to introduce the economics profession to the spatial EBCMs with heat transport as a potentially useful alternative for studying the economics of climate change relative to the simple carbon cycle models. By deriving the spatiotemporal profile for optimal taxes from the one-dimensional coupled climate economic model, we show how the spatial EBCMs can contribute to the current debate regarding how much to mitigate now, whether mitigation policies should be spatially homogeneous or not, and how to derive geographically specific information regarding dam-

ages and policy measures.[3]

A popular class of EBCMs which we focus upon, are the models of North (North 1975 a, 1075b, North et al. 1981, Wu and North 2007).[4] A common feature among these models is (i) the explicit incorporation of the spatial dimension into the climate model in the form of heat diffusion or transport across latitudes, and (ii) the spatial dependency of earths albedo due to the presence of an endogenous ice line where latitudes north (south) of the ice line are solid ice and latitudes south (north) of the ice line are ice free.

Since these models are new in economics we proceed in steps that we believe make this methodology accessible to economists. In section 2 we present a basic energy balance climate model[5] which incorporates human impacts on climate which result from carbon dioxide accumulation due to the use of fossil fuels, that blocks outgoing radiation. In developing the model we follow North (1975 a,b) and use his notation. We use the model to expose solution methods and especially the two mode approach which transforms systems of partial differential equations (PDEs) in infinite dimensional spaces resulting from the spatial modeling, into systems of ordinary differential equations

---

[3]Another issue that can be addressed by latitude dependent climate models is damage reservoirs. Damage reservoirs in the contest of climate change can be regarded as sources of climate damages which will eventually cease to exit when the source of the damages is depleted. Damage reservoirs are latitude dependent and ice lines and permafrost can be regarded as such reservoirs.

As the ice lines move closer to the poles, due to climate change, we might expect that marginal damages from this moving will be large at first and then diminish as the ice line approaches the Poles. When there will be no ice left at the Poles this damage reservoir would have been exhausted. The presence of an endogenous ice line in the EBCM allows us to model these type of damages explicitly given the relevant information

Permafrost is soil at or below the freezing point of water for two or more years. The permafrost feedback suggests that permafrost carbon emissions could affect long-term projections of future temperature change. Studies indicate that up to 22 % of permafrost could be thawed already by 2100. Once unlocked under strong warming, thawing and decomposition of permafrost can release amounts of carbon until 2300 comparable to the historical anthropogenic emissions up to 2000 (approximately 440 GTC) (Schneider et al. 2011).

EBCMs by explicitly introducing the spatial dimension in the climate module of the problem can help in the understanding of these type of latitude dependent damages which may have an important effect on the temporal and spatial structure of policy instruments, because of the 'front loading' character of damages and the possible relations with tipping points and thresholds.

Judd and Lontzek (2011) have formulated a dynamic stochastic version of DICE - the SDICE - which includes stochastic tipping points possibilities. They show that this complexity affects the optimal policy results in comparison to RICE. The modeling of damage reservoirs is beyond the purpose of this paper, but we think that it represents an important area for further research.

[4]Although the EBCMs that we use are simple climate models, many useful insights into climate dynamics can arise from these simple models (Pierrehumbert 2008).

[5]For more on EBCMs see for example Pierrehumbert (2008) (chapters 3 and 9, especially sections 9.2.5 and 9.2.6 and surrounding material). North et al (1981) is a very informative review of EBCMs while Wu and North (2007) is a recent paper on EBCMs.

(ODEs) in finite dimensional spaces. The two mode approach will be used to solve, and numerically approximate latitude dependent temperature and damage functions.

Section 3 couples the spatial EBCM with an economic growth model, where a finite stock of fossil fuel is an essential input along with capital and labour. Fossil fuels are extracted by fossil fuel firms which pay taxes on profits and used by firms producing consumption goods which pay per unit of fossil fuel used. We solve the model for the social planner and for the competitive equilibrium with taxes. We derive the optimal taxes and their spatial structure and their temporal profiles. We show that if international transfers among regions are not possible, poor regions should pay relatively lower fossil fuel taxes, and that under a mild assumption about a slow decay of the CO_2 in the atmosphere the profit tax on fossil fuel firms decline over time and the unit taxes on extracted fossil fuels grow at rate less than the rate of return on capital. Furthermore we derive the latitude dependent temperature function and the impact of heat transport on damages across latitudes.

In section 4 we use approximate solutions, we simulate the climate and the economic model and we derive explicit numerical solutions for the latitude dependent temporal and damage functions. The last section concludes.

2   An Energy Balance Climate Model with Human Inputs

In this section we develop a one-dimensional Energy Balance Climate Model with human inputs. The term "one-dimensional" means that there is an explicit one dimensional spatial dimension in the model so that our unified model of the climate and the economy evolves both in time and space.[6] We follow North (1975a,b) and North et al. (1981) in this development.

Let x to denote the *sine* of the latitude. We shall abuse language and just refer to x as "latitude". Following North (1975a,b) let $I(x,t)$ denote outgoing infrared radiation flux measured in W/m^2 at latitude x at time t, $T(x,t)$ denote surface (sea level) temperature measured in °C at latitude x at time t. The outgoing radiation and surface temperature can be related

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[6]In contrast, a "zero-dimensional "model does not explicitly account for the spatial dimension. On the other hand more complicated spatial structures could include two-dimensional spherical models. Our methods can be readily applied to a two dimensional spherical worlds as in Wu and North (2007).

through the empirical formula.[7]

$$I(x,t) = A + BT(x,t), \ A = 201.4W/m^2, \ B = 1.45W/m^2 \qquad (1)$$

The basic energy balance equation developed in North (1975a, equation (29)) can be written, with human input added, as:

$$\frac{\partial I(x,t)}{\partial t} = QS(x)\alpha(x,x_s(t)) - [I(x,t) - h(x,t)] + D\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial I(x,t)}{\partial x}\right] \qquad (2)$$

where units of x are chosen so that $x = 0$ denotes the Equator, $x = 1$ denotes the North Pole, and $x = -1$ denotes the South Pole; Q is the solar constant[8] divided by 2; $S(x)$ is the mean annual meridional distribution of solar radiation which is normalized so that its integral from -1 to 1 is unity; $\alpha(x,x_s(t))$ is the absorption coefficient or co-albedo function which is one minus the albedo of the earth-atmosphere system, with $x_s(t)$ being the latitude of the ice line at time t; and D is a thermal diffusion coefficient that has been computed as $D = 0.649Wm^{-2\circ}\text{C}^{-1}$ (North at al.1981)

Equation (2) states that the rate of change of outgoing radiation is determined by the difference between the incoming absorbed radiant heat $QS(x)\alpha(x,x_s(t))$ and the outgoing radiation $[I(x,t) - h(x,t)]$. Note that the outgoing radiation is reduced by the human input $h(x,t)$. Thus the human input at time t and latitude x, can be interpreted as the impact of the accumulated carbon dioxide that reduces outgoing radiation.

We define $h(x,t) = \sigma(x)\xi\ln\frac{M(t)}{M_0}$ where M_0 denotes the preindustrial and $M(t)$ the time t stock of carbon dioxide in the atmosphere, $\xi = 5.35$ (IPCC 2001) is a temperature-forcing parameter (°C per$W$ per m^2), and $\sigma(x)$ is a weighting function that capture latitudinal differences in the impact of the stock of the atmospheric carbon dioxide on latitude x's temperature via (1) and (2). The stock of the atmospheric carbon dioxide evolves according to

$$\dot{M}(t) = \int_{x=-1}^{x=1} \beta(x,t)q(x,t)dx - mM(t), \ M(0) = M_0 \qquad (3)$$

where $\beta(x,t)q(x,t)$ are emissions generated at latitude x, with emissions being proportional to the amount of fossil fuels used by latitude x at time

---

[7]It is important to note that the original Budyko (1969) formulation cited by North parameterizes A,B as functions of fraction cloud cover and other parameters of the climate system. North (1975b) points out that due to non-homogeneous cloudiness A and B should be functions of x. There is apparently a lot of uncertainty involving the impact of cloud dynamics (e.g. Trenberth et al. 2010 versus Lindzen and Choi 2009). Hence robust control in which A,B are treated as uncertain may be called for but this is left for further research.

[8]The solar constant includes all types of solar radiation, not just the visible light. It is measured by satellite to be roughly 1.366 kilowatts per square meter (kW/m^2).

t. The coefficient $\beta(x,t)$ reflects emission intensity of the fossil fuels used at latitude x and its dependence on time represents the possibility of technical progress in emission intensity, finally m is the carbon decay rate.

We assume that the total stock of fossil fuel available is fixed or,

$$\int_{x=-1}^{x=1} q(x,t)\,dx = q(t) \ , \ \int_0^\infty q(t) = R_0 \tag{4}$$

where $q(t)$ is total fossil fuels used across all latitudes at time t, and R_0 is the total available amount of fossil fuels on the planet. Thus in this model use of fossil fuels generates emissions, emissions increase the stock of atmospheric carbon dioxide, which in turn increases the temperature by blocking the outgoing radiation.

As pointed out by North (1975b), in equilibrium the incoming absorbed radiant heat at a given latitude is not matched by the net outgoing radiation and the difference is made by the meridional divergence of heat flux which is modelled by the term $D\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial I(x,t)}{\partial x}\right]$. This term explicitly introduces the spatial dimension stemming from the heat transport, into the climate model.

Returning to the description of (2), the ice line is determined dynamically by the condition (Budyko 1969, North 1975 a,b):

$$\begin{array}{ll} T > -10°\text{C} & \text{no ice line present at latitude } x \\ T < -10°\text{C} & \text{ice present at latitude } x \end{array} \tag{5}$$

and 'below' the ice line absorption drops discontinuously because the albedo jumps discontinuously. For example North (1975a) specifies, discontinuous co-albedo function as:

$$\alpha(x, x_s) = \left\{ \begin{array}{ll} \alpha_0 = 0.38 & |x| > x_s \\ \alpha_1 = 0.68 & |x| < x_s \end{array} \right. \tag{6}$$

2.1 Approximating Solutions for the Basic Energy Balance Equation

We turn now to a more detailed analysis of the solution process. Equation (2) is a PDE. One might think that we are going to have to deal with the complicated mathematical issues of the solution or the optimal control of PDEs when we need to discuss the economic optimization problems over space and time. But, as we shall see, the climate problem reduces to the optimal control of a small number of "modes" where each "mode" follows a simple ODE. We believe this decomposition is another important and new contribution of our paper to the study to coupled economic and climate models. Let us continue with the development of the solution procedure for equation (2) before turning to optimization.

North (1975b) approached the solution of (2) by using approximation methods.[9] In this case the solution is approximated as $\hat{I}(x,t) = \sum_{n \text{ even}} I_n(t)P_n(x)$, where $I_n(t)$ are solutions to appropriately defined ODEs and $P_n(x)$ are even numbered Legendre polynomials. A satisfactory approximation of the solution for (2) can be obtained by the so called two mode solution where $n = \{0,2\}$. We develop here a two mode solution given the human forcing function $h(x,t)$. Since we are going to use the temperature as the basic state variable we redefine (2) using (1), in terms of temperature $T(x,t)$ and we have

$$
\begin{aligned}
B\frac{\partial T(x,t)}{\partial t} &= QS(x)\alpha(x,x_s) - [(A + BT(x,t)) - h(x,t)] + \quad (7) \\
&\quad DB\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial T(x,t)}{\partial x}\right]
\end{aligned}
$$

Using the approximation $\hat{T}(x,t) = \sum_{n \text{ even}} T_n(t)P_n(x)$, where now $T_n(t)$ are solutions to appropriately defined ODEs the two mode solution is defined as:

$$
\hat{T}(x,t;D) = T_0(t) + T_2(t;D)P_2(x) \tag{8}
$$

$$
\begin{aligned}
B\frac{dT_0(t)}{dt} &= -A - BT_0(t) + \\
&\quad \int_{-1}^{1}\left[QS(x)\alpha(x,x_s) + \xi\ln\frac{M(t)}{M_0}\sigma(x)\right]dx \tag{9}
\end{aligned}
$$

$$
\begin{aligned}
B\frac{dT_2(t)}{dt} &= -B(1+6D)T_2(t) + \\
&\quad \frac{5}{2}\int_{-1}^{1}\left[QS(x)\alpha(x,x_s) + \xi\ln\frac{M(t)}{M_0}\sigma(x)\right]P_2(x)dx \tag{10}
\end{aligned}
$$

$$
T_0(0) = T_{00}, T_2(0) = T_{20}, P_2(x) = \frac{(3x^2 - 1)}{2} \tag{11}
$$

$$
S(x) = 0.5\left[1 + S_2 P_2(x)\right] \ , \ S_2 = -0.482 \tag{12}
$$

The derivation of the solution is presented in Appendix 1.[10] Given the definitions of the functional forms the two mode solution is tractable and can be calculated given initial conditions T_{00}, T_{02} which are determined by the initial climate state.

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[9] For a general approach to approximation methods see for example Judd (1998).

[10] The two mode solution is an approximating solution. We can develop a series of approximations of increasing accuracy by solving this problem for expansions using a "two mode" solution, a "three mode" solution and so on. North's results suggest that the two mode solution is an adequate approximation for nonoptimizing models. We use the two-mode approximation in our optimal control setting. A topic of further research could be an investigation of how many modes are needed for a good quality approximation in an optimal control setting.

In the two-mode solution, the ice line function $x_s(t)$ which determines the co-albedo solves the equation $I_s = I(x_s(t), t)$. In terms of temperature and using the two-mode solution, the ice line function solves

$$\hat{T}(x, t; D) = T_0(t) + T_2(t; D)P_2(x_s(t)) = T_s, \ T_s = -10°C \qquad (13)$$

and the ice line function is given by a solution of (13), i.e.

$$x_s(t) = P_+^{-1}\left(\frac{T_s - T_0(t)}{T_2(t; D)}\right) \qquad (14)$$

Where the subscript "+" denotes the largest inverse function of the quadratic function $P_2(x) := (1/2)(3x^2 - 1)$. Notice that the inverse function is unique and is the largest one on the set of latitudes $[-1, 1]$. Thus there exist a nonlinear feedback from changes in temperature to the co-albedo thought the endogeneity of the ice line. This feedback can be simplified by making the co-albedo function $\alpha(x, x_s)$ a smooth function of the temperature, $\alpha\left(x, \hat{T}(x, t; D)\right)$ which can be highly nonlinear around $-10°C$.[11] A more simplified and tractable specification of the co-albedo is the one introduced by North et al. (1981, p.95 equation (18), where the co-albedo depends only on geographical location or

$$a(x) = 0.681 - 0.202P_2(x) \qquad (15)$$

In this case the co-albedo function retains its latitude dependence and provides a significant simplification that helps tractability.

2.1.1 Use of global mean temperature and potential bias

The two-mode solution defines the climate module by (8)-(12), and (3),(4). Although the climate module does not contain the PDE (7) that incorporates temperature diffusion, spatial interactions are incorporated through the mode-2 part of the solution the ODE (10). Thus the contribution of the second mode into the full solution can be regarded as the "importance of space" through heat transport, in the analysis of climate change. This can be seen by the following argument.

The size of diffusion coefficient D determines the speed of spatial diffusion in (7). If $D = 0$ then are no spatial interactions, if $D \to \infty$ then we have instantaneous mixing and spatial homogeneity and thus the heat transport across latitudes is not relevant for our problem. In this case, the mode two solution vanishes. To show this note that since the total amount of fossil fuel is finite and the contributions to the stock of atmospheric carbon dioxide is due to the use of fossil fuels, the stock of the atmospheric carbon dioxide $M(t)$ must be bounded above. Thus the second term of the right hand side of (10) is bounded above. Then the following proposition can be stated

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[11]For example the co-albedo function $\alpha(x, T(x, t)) = c_0 + c_1 \tanh(T(x, t) + 10)$ for $(c_0, c_1) = (.525, .195)$ provides a good approximation of the discontinuous function (5).

Proposition 1 *Assume that $\int_{-1}^{1}\left[QS(x)\alpha(x,x_s)+\xi\sigma(x)\ln\frac{M(t)}{M_0}\right]P_2(x)dx = \Phi(t) \leq UB < \infty$, and that $D \rightarrow \infty$. Then the solution $T_2(t)$ of (10) vanishes.*

For the proof see Appendix 2. Thus for a given diffusion $D < \infty$ the relative contribution of $T_2(t)$ to the solution $\hat{T}(t)$ can be regraded as an a measure of whether the heat transport is important in the solution of the problem.

This result can be used to suggest that the use of the global mean temperature alone in IAMs may introduce a bias. From the two mode approximation of the temperature, we obtain the global mean temperature as $m_T = T_0(t)$.[12] This result, along with proposition 1, indicates that the zero - dimensional IAMs can be regarded as a special case of a one-dimensional model when $D \rightarrow \infty$. Thus the second mode that provides that spatial distribution of temperature is omitted in the zero - dimensional IAMs. Since scientific evidence indicate that D is small (less than one according to North et al. 1981) our result suggest that omitting the second mode introduces a bias. In our paper we correct for this underlying bias by keeping that second mode, and we also provide a basis for a quantitative representation of this bias. The variance of the global mean temperature is:

$$V_T = \int_{-1}^{1}\left[\hat{T}(x,t;D)-T_0(t)\right]^2 dx = \int_{0}^{1}(T_2(t;D)P_2(x))^2 dx = \frac{2}{5}(T_2(t;D))^2 \tag{16}$$

In an IAM this variance will be zero since the second mode is dropped.

Local temperature means at latitudes $(x, x+dx)$ and the mean of temperature over the set of latitudes $Z = [a,b]$ are defined by

$$[T_0(t)+T_2(t;D)P_2(x)]\,dx, m[a,b]=\int_{a}^{b}[T_0(t)+T_2(t;D)P_2(x)]\,dx \tag{17}$$

while the variance of temperature over the set of latitudes $Z = [a,b]$ is

$$V[a,b]=\int_{a}^{b}[T_0(t)+T_2(t;D)P_2(x)-m[a,b;t]]^2\,dx \tag{18}$$

It might be plausible to assume that utility in each area $[a,b]$ depends upon both the mean temperature and the variance of temperature in that area. For example we may expect increases in mean temperature and variance to have negative impacts on output in any area Z, if it is located in tropical latitudes. Whereas mean temperature increases in some areas Z (e.g. Siberia) may increase utility rather than decrease utility.[13]

---

[12]This is because $m_T = \int_{-1}^{1}\hat{T}(x,t)dx = \int_{-1}^{1}[T_0(t)+T_2(t)P_2(x)]\,dx$ and $\int_{-1}^{1}P_2(x)dx = 0$.

[13]In a stochastic generalization of our model, we could introduce a stochastic process to

3   An Economic EBC Model

3.1   Potential world output and damages from climate change

Output at each location of our economy is produced according to a standard neoclassical production function which is assumed to be of the Cobb Douglas form with constant returns to scale and exponentially growing total factor productivity, or

$$
\begin{aligned}
Y\left(t,x\right) &= \mathbb{A}(x,t)\Omega(T(x,t))F(K(x,t),L(x,t),q(x,t)) \\
&= e^{at}\mathbb{A}(x,0)\Omega(T(x,t))K(x,t)^{\alpha_K}L(x,t)^{\alpha_L}q(x,t)^{\alpha_q} \\
&= e^{(a+n\alpha_L)t}\mathbb{A}(x,0)L(x,0)^{\alpha_L}\Omega(T(x,t))K(x,t)^{\alpha_K}q(x,t)^{\alpha_q} \\
&\equiv e^{(a+n\alpha_L)t}\Psi(x,T(x,t))K(x,t)^{\alpha_K}q(x,t)^{\alpha_q}
\end{aligned}
\tag{19}
$$

where $K(x,t), L(x,t), q(x,t)$ denote capital, labour and fossil fuels respectively used at latitude (location) x, and time t, a is TFP growth, n is population growth, and $\Omega(T(x,t)$ are damages to output due to climate change at latitude x and time t as a function of temperature at the same latitude, with $\frac{\partial \Omega(T(x,t))}{\partial T} < 0$.

In this economy we define by $F_{total}(K(t),q(t),\{T(x,t)\}_{x=-1}^{x=1};t)$ the "*potential world GDP at date t*". This concept represents the maximum output that the whole world can produce given total world capital $K(t)$ available and total world fossil fuel $q(t)$ used, for a given distribution of temperature $T(x,t)$ across the globe, with labor growing at a constant rate n, and treated as realistically immobile.[14] Thus F_{total} can be regarded as a natural base line under ideal world conditions where there's no barriers to capital and fossil fuel flows to their most productive uses across latitudes.[15] We abuse notation and write $F_{total}(K(t),q(t),\{T(x,t)\}_{x=-1}^{x=1};x,t) = F_{total}(K(t),q(t),T;t)$. The overall resource constraint for the economy can then be defined as:

$$
C(t) + \dot{K}(t) + \delta K(t) = F_{total}(K(t),q(t),T;t)
\tag{20}
$$

where total consumption, capital and fossil fuel are defined over all latitudes as: $j\left(t\right) = \int_{x=-1}^{x=1} j\left(x,t\right)dx$, $j = C,K,q$ respectively. The potential world GDP can be analytically defined in the following way:

---

represent "weather," i.e. very high frequency fluctuations relative to the time scales we are modeling here. Here the "local variance" of high frequency phenomena like "weather" may change with changes in lower frequency phenomena such as mean area Z temperature and area Z temperature variance. We leave this task to future research. Existing dynamic integrated models of climate and economy, (e.g. Nordhaus's well known work (2007, 2010)) can not deal with these kinds of spatial elements, such as impacts of changes in temperature variance, generated by climate dynamics over an area Z.

[14]Labor immobility at a global scale could be regarded as a reasonable approximation given restrictions on labor mobility relative to capital and fossil fuel mobility.

[15]This notion can be regarded as similar to the notions of "potential GDP" "potential output" etc used by macroeconomists.

Using $\Psi(x, T(x, t))$ from (19), define damages at a specific location x and global damages as:

$$J(x, t; D) = \frac{\Psi(x, T(x, t))^{1/\alpha_L}}{\left[\int_{x'} \Psi(x', T(x', t))^{1/a_L} dx'\right]^{a_K + a_q}} \tag{21}$$

$$J\left(\{T(x, t)\}_{x=-1}^{x=1}\right) = J(t; D) \equiv \int_x J(x, t; D) \, dx \tag{22}$$

respectively. Potential world GDP $F_{total}(K(t), q(t), T; t)$, can be computed through the following optimization problem:

$$F_{total}(K(t), q(t), T; x.t)$$
$$\equiv \max\{\int_x e^{(a+n\alpha_L)t} \Psi(x, T(x, t)) K(x, t)^{\alpha_K} q(x, t)^{\alpha_q} dx, \tag{23}$$
$$s.t. \int_x K(x, t) dx \leq K(t), \int_x q(x, t) dx \leq q(t)\}.$$

The Lagrangian associated (23) is:

$$\mathcal{L} = \int_x e^{(a+n\alpha_L)t} \Psi(x, T(x, t)) K(x, t)^{\alpha_K} q(x, t)^{\alpha_q}) dx + \tag{24}$$

$$\mu_K(t)\left[K(t) - \int_X K(x, t) dx\right] + \mu_q(t)\left[q(t) - \int_X q(x, t) dx\right] \tag{25}$$

which leads to

$$a_K e^{(a+n\alpha_L)t} \Psi(x, T(x, t)) K(x, t)^{\alpha_K - 1} q(x, t)^{\alpha_q} = \mu_K(t) \tag{26}$$
$$a_q e^{(a+n\alpha_L)t} \Psi(x, T(x, t)) K(x, t)^{\alpha_K} q(x, t)^{\alpha_q - 1} = \mu_q(t) \tag{27}$$

which means that the marginal product of capital and the marginal product of the fossil fuels are equated across latitudes for all times t, in the context of the potential world GDP notion. Furthermore, since F is Cobb-Douglas it follows that

$$K(x, t) = [\Psi(x, T(x, t; D))^{1/\alpha_L} / \int_{x'} \Psi(x', T(x', t))^{1/a_L} dx'] K(t) \tag{28}$$

$$q(x, t) = [\Psi(x, T(x, t; D))^{1/\alpha_L} / \int_{x'} \Psi(x', T(x', t))^{1/a_L} dx'] q(t) \tag{29}$$

$$F_{total}(K(t), q(t), T; t) = \left[e^{(a+\alpha_L n)t} K(t)^{\alpha_K} q(t)^{\alpha_q}\right] J(t; D) \tag{30}$$

As it can be seen from (30) the Cobb-Douglas specification allows the "separation" of the climate damage effects on production across latitudes, as the "index" $J(t; D)$, which depends on thermal diffusion coefficient D, multiplies a production function that is independent of x. Thus population growth and technical change affect the "macrogrowth component" $e^{(a+\alpha_L n)t} K(t)^{\alpha_K} q(t)^{\alpha_q}$, while changes in the size of D have a direct effect

on the "climate component". The combination of the macrogrowth and the climate component determine the potential world input. This separability property allows a more tractable analytical and numerical work regarding the importance of the spatial dimension in the economic-climate model.

From the consumer side, the idea of working at the global scale suggests a welfare optimization problem that can be interpreted as the maximization of the welfare of an "aggregate dynastic consumer family" subject to an aggregate production function. This problem can be set in the following way:. Allocate $C(t)$ to solve the problem

$$\max\{\int_x (e^{(-\rho+n(1-\gamma))t}L(x,0)^{1-\gamma}\frac{C(x,t)^\gamma}{\gamma})dx, \ \int_x C(x,t)dx \leq C(t)\} \quad (31)$$

to obtain:

$$C(x,t) = \frac{L(x,0)}{\int_x L(x,0)dx}C(t) \quad (32)$$

Allowing for per capita damages in utility due to climate change given by $\Omega_C(T(x,t))$, with $\partial\Omega_C(T(x,t))/\partial T > 0$. the economic part of the social welfare problem in the "Ramsey-like" form for the "aggregate dynastic consumer family" can be written as:,

$$\max \int_{t=0}^\infty e^{(-\rho+n(1-\gamma))t}\frac{C(t)^\gamma}{\gamma}\int_x\left[\frac{L(x,0)}{\int_{x'}L(x',0)dx'}\right]^\gamma dxdt -$$
$$\left[\int_{t=0}^\infty e^{(-\rho+n)t}\int_x L(x,0)\Omega_C(T(x,t)dx\right]dt, \quad (33)$$

subject to

$$C(t) + \dot{K}(t) + \delta K(t) = e^{(a+\alpha_L n)t}K(t)^{\alpha_K}q(t)^{\alpha_q}J(T;D) \quad (34)$$

$$\int_{t=0}^\infty q(t)dt \leq R_0 \quad (35)$$

with R_0 denotes the total available amount of fossil fuel on the planet.

3.2 Global welfare maximization

Given the economy described above and the climate described by the EBCM, we analyze the welfare maximization problem considered by a social planner. This economic part of this problem is defined in terms of the potential world GDP and the aggregate dynastic consumer family as

$$\max \int_0^\infty e^{-\rho t}\int_x \upsilon(x) L(x,t)\left[U\left(\frac{C(x,t)}{L(x,t)}\right) - \Omega_C(T(x,t))\right] dxdt \quad (36)$$

subject to (20),(35),(3),(7) the total consumption and total fossil fuel constraints, along with the appropriate initial conditions, where $\upsilon(x)$ are exogenously given welfare weights such that $\int_{x=-1}^{x=1}\upsilon(x)\,dx = 1$. Varying the

weights all the Pareto efficient allocations can be traced. A standard weight independent of x for our case is $\upsilon = 1$. Assuming zero extraction cost for the fossil fuels,[16] denoting by λ costate variables by μ Lagrangian multipliers and assuming $\beta(x,t) = \beta$ to avoid corner solutions, the current value Hamiltonian for this problem can be written as:

$$
\begin{aligned}
\mathcal{H} \;=\; & \int_X \upsilon(x) L(x,t) \left[U\left(\frac{C(x,t)}{L(x,t)} \right) - \Omega_C(T(x,t)) \right] dx + \\
& \lambda_K(t) \left[F_{total}(K(t), q(t), T; t) - C(t) - \delta K(t) \right] \\
& + \mu_R \left[R_0 - \int_0^\infty q(t) \right] + \lambda_M(t) \left[\int_{-1}^1 \beta(t) q(x,t)\, dx - m M(t) \right] \\
& + \lambda_T(t,x) \left[\frac{1}{B} \left[QS(x)\alpha(x, T(x,t)) - (A + BT(x,t)) \right. \right. \\
& \left. + \sigma(x) \xi \ln \frac{M(t)}{M_0} + DB \frac{\partial}{\partial x} \left[(1 - x^2) \frac{\partial T(x,t)}{\partial x} \right] \right] \\
& \left. + \mu_C(t) \left[C(t) - \int_X C(x,t)\, dx \right] + \mu_q(t) \left[q(t) - \int_X q(x,t) dx \right] \right]
\end{aligned}
\tag{37}
$$

In this problem the state variables are $\mathbf{v} = (K(t), R(t), M(t), T(t,x))$, while we use as control $\mathbf{u} = (C(t), C(x,t), q(t), q(x,t))$

The maximum principle implies for the controls:[17]

$$
C(t), C(x,t) \quad : \quad \lambda_K(t) = \mu_C(t) = \upsilon(x) U'\left(\frac{C(x,t)}{L(x,t)} \right) \tag{38}
$$

$$
q(t) \quad : \quad \lambda_K(t) F'_{total,q} = \mu_R - \mu_q(t) \tag{39}
$$

$$
q(x,t) \quad : \quad \lambda_M(t) \beta(t) = \mu_q(t) \tag{40}
$$

$$
\text{or } F'_{total,q} \;=\; \frac{\mu_R - \lambda_M(t) \beta(t)}{\lambda_K(t)}, \tag{41}
$$

For weights independent of x (38) implies that per capital consumption should be equated across locations. For the costates we have:

---

[16] This simplifying assumption does not affect the validity of our results.

[17] Since problem (36) is non autonomous, we assume that the discount rate sufficiently high and that the functions of the problem satisfy the growth conditions required to apply the Pontryagin maximum principle (Malysh 2008). To ease notation sometimes we denote derivatives by the subscript for the relevant variable and a $(')$.

$$\dot\lambda_K(t) = \left[\rho + \delta - F'_{total,K}(K(t), q(t), T; t)\right] \lambda_K(t) \tag{42}$$

$$\dot\lambda_M(t) = (\rho + m) \lambda_M(t) - \frac{\xi}{BM(t)} \int_{-1}^{1} \sigma(x) \lambda_T(t, x) \tag{43}$$

$$\dot\lambda_T(t, x) = (\rho + 1) \lambda_T(t, x) + \upsilon(x) L(t, x) \Omega'_{c,T}(T(t, x)) \tag{44}$$
$$-\lambda_K(t) F'_{total,T}(K(t), q(t), T; t) -$$
$$QS(x) \frac{\lambda_T(t, x)}{B} \frac{\partial\alpha(x, T(x, t))}{\partial T} - D\frac{\partial}{\partial x}\left[(1 - x^2)\frac{\partial\lambda_T(x, t)}{\partial x}\right]$$

The last term on the RHS of (44) is obtained by applying the maximum principle. It requires, in the derivation of the conditions of the maximum principle, to differentiate by part twice with respect to x, in order to express the derivatives of T with respect to x in terms of derivatives of λ_T with respect to x. A detailed argument is presented in Appendix 3.

A solution of the welfare maximization problem, provided it exists and satisfies the desirable stability properties, will determine the optimal temporal and latitudinal paths for the states, the controls and the costates. From (44) and the dependency of the solutions of the dynamical system on parameters it follows that the optimal time paths will be dependent on the thermal diffusion coefficient D. Denoting optimality by a ($*$) these paths can be written as:

$$\{K^*(t; D), K^*(t, x; D) R^*(t; D), M^*(t; D), T^*(t, x; D)\}_{x=-1}^{x=1}$$
$$\{C^*(t; D), C^*(x, t; D), q^*(t; D), q^*(x, t; D)\}_{x=-1}^{x=1}$$
$$\{\lambda_K^*(t; D), \lambda_R^*(t; D), \lambda_M^*(t; D), \lambda_T^*(t, x; D)\}_{x=-1}^{x=1} \tag{45}$$

Substituting these paths into (21) and (22) will determine the "optimal" damages from climate change on a global or on a location basis.

3.3 Market Equilibrium with Fossil Fuel Taxes

The model presented in the previous section is general enough for studying optimal taxation problems in a general equilibrium setup, it requires however working with systems of PDEs in infinite dimensional spaces. In order to provide tractable results we use this model as a basis but we simplify the climate dynamics by using the two-mode approximation developed in section 2.1. To study the optimal taxation problem we consider a global market economy with each latitude x considered as a country. In each country the representative consumer maximizes utility subject to a permanent income constraint by considering as parametric damages due to climate change, the representative firm maximizes profits by considering as parametric fossil fuel world prices and taxes on fossil fuel use. World fossil fuel firms maximize

profits by considering as parametric taxes on their profits. The central planner, given the optimizing choices of firms and consumers, chooses the amount of fossil fuel $q(x, \cdot)$ to be used in each location, by taking onto account climate change damages and climate dynamics. The choice of fossil fuel allocation determines optimal taxes.

3.3.1 Consumers

Consumers at latitude (or country) x are a "dynastic family" that takes $\Omega_C(T(x,t))$ as parametric beyond their control and can borrow and lend on world bond markets at the rate $r(t)$ to solve the following problem[18]

$$\max_{\{C(x,t)\}} \{ \int_{t=0}^{\infty} e^{-\rho t} L(x,t) U\left(\frac{C(x,t)}{L(x,t)}\right) - \Omega_C(T(x,t;D))dt\} \qquad (46)$$

subject to

$$C(x,t) + \dot{K}(x,t) + \dot{B}(x,t) = r(t)(K(x,t) + B(x,t)) + I(x,t) \quad (47)$$

$$B(x,0) = 0, K(x,0) = K_0(x) \qquad (48)$$

$$I(x,t) \equiv w(x,t)L(x,t) + s_{FF}(x,t)\pi_{FF}(t) + s_{Tax}(x,t)Tax(t) \quad (49)$$

where $B(x,t)$ denote bonds held at location x and time t. After tax profits from fuel firms are redistributed lumpsum to latitude x consumers in the fraction $s_{FF}(x,t)$ and proceeds from fuel taxes are redistributed lump sum to latitude x consumers in the fraction $s_{Tax}(x,t)$.[19] Set $R(t) = \int_{s=0}^{t} r(s)ds$ and multiply both sides of (47) by $\exp(-R(t))$, integrate over t, and impose the "solvency" constraints

$$B(x,t)e^{-R(t)} \to 0, \quad K(x,t)e^{-R(t)} \to 0 \quad ,t \to \infty. \qquad (50)$$

Observe that we can write (47) in the present discounted value form

$$\int_{t=0}^{\infty} e^{-R(t)} C(x,t)dt = K_0(x) + \int_{t=0}^{\infty} e^{-R(t)} I(x,t)dt \qquad (51)$$

which leads to the FONC

$$U'\left(\frac{C(x,t))}{L(x,t}\right) = \Lambda(x)e^{(\rho t - R(t))} \qquad (52)$$

where $\Lambda(x)$ is the Lagrange multiplier for the permanent income constraint (51) and expresses the marginal utility of capitalized income at location x.

---

[18]One might think that one could put the term $+\delta K(x,t)$ in (47) but this will lead immediately to bonds crowding out capital or other kinds of corner solution problems. Therefore we treat $+\delta K(x,t)$ as an expense paid by firms on the firm side of the model.

[19]In baseline analysis using Arrow Debreu private ownership economies, it is standard to assume perfect markets (borrowing and lending with no frictions, defaults, etc.) with profits and taxes redistributed lump sum to consumers.

Note that per capita consumptions will not be equated across latitudes unless $\Lambda(x) = \Lambda(x')$ for all x, x'. For equality of the marginal utility across latitudes we will need to assume, following the theory of the Second Welfare Theorem, that intertemporal endowment flows are adjusted so that $\Lambda(x) = \Lambda(x')$ for all x, x'.

By setting $\dot{B} = Z$, The consumer optimum can be alternatively characterized by the following Hamiltonian function.

$$
\begin{aligned}
H^c &= L(x,t)(U(C(x,t)/L(x,t)) - \Omega_C(T(x,t;D)) + \\
&\quad \lambda_K(x,t)\left[r(t)(K(x,t) + B(x,t)) + I(x,t) - C(x,t) - Z(x,t)\right] + \\
&\quad \lambda_B(x,t) Z(x,t)
\end{aligned} \tag{53}
$$

with optimality conditions

$$
\begin{aligned}
(U'(C(x,t)/L(x,t)) &= \lambda_K(x,t) = \lambda_B(x,t) \tag{54} \\
\dot{\lambda}_K(x,t) &= (\rho - r(t))\lambda_K(x,t) \tag{55}
\end{aligned}
$$

along with transversality conditions. Thus, competitive equilibrium with borrowing and lending forces leads to:

$$
\dot{\lambda}_K(x,t)/\lambda_K(x,t) = \rho - r(t) = \dot{\lambda}_K(x',t)/\lambda_K(x',t) \tag{56}
$$

for each x, x' for all t, where $\lambda_K(x,t)$ is the current shadow value of capital at location x. However this is not enough to force $\lambda_K(x,t) = \lambda_K(x',t)$ for each x, x', and the implied equation of per capita consumption across latitudes, for all dates t, unless intertemporal endowment flows are adjusted.

3.3.2 Consumption Goods Producing Firms

Firms producing consumption goods located at latitude x solve the problem

$$
\begin{aligned}
\max \ \{&\mathbb{A}(x,t)\Omega(T(x,t))F(K(x,t), L(x,t), q(x,t)) - \\
&(r(t) + \delta)K(x,t) - w(x,t)L(x,t) - (p(t) + \tau(x,t))q(x,t)\}
\end{aligned} \tag{57}
$$

where $p(t)$ is the world price for fossil fuels, $w(x,t)$ is the wage at location x and time t,[20] and $\tau(t,x)$ is a tax on fossil fuels paid by the representative firm located at point (country) x, and $F(K, L, q)$ is constant returns to scale. Hence profits will be zero at each x for firms that produce consumption goods. The optimality conditions for the optimal choices for K and q imply:

$$
\begin{aligned}
\mathbb{A}(x,t)\Omega(T(x,t;D))F_K'(K(x,t), L(x,t), q(x,t)) &= r(t) + \delta \tag{58} \\
\mathbb{A}(x,t)\Omega(T(x,t;D))F_q'(K(x,t), L(x,t), q(x,t)) &= p(t) + \tau(x,t) \tag{59}
\end{aligned}
$$

---

[20] Wages are not equated across locations due to labour immobility.

Thus in any decentralized problem latitude x firms will choose demands $K(x,t)$ and $q(x,t)$ according to (58) and (59). Note that these marginal value products are equated across $x's$ for every date t only if taxes on fossil fuels are equal across locations or $\tau(x,t) = \tau(t)$.

From (55) and (58) it follows that in market equilibrium

$$\dot{\lambda}_K(x,t) = \left(\rho + \delta - F_K'\right)\lambda_K(x,t) \tag{60}$$

3.3.3 Fossil fuel firms

World fossil fuel firms solve the problem

$$\max_{q(t)} \int_{t=0}^{\infty} \exp\left[-\int_{s=0}^{t} r(s)ds\right][(p(t)q(t)(1-\theta(t))])dt, \tag{61}$$

$$\text{subject to} \quad \int_{t=0}^{\infty} q(t)dt \leq R_0 \tag{62}$$

where, $\theta(t)$ denotes profit tax on fossil fuel firms. Let μ_0 denote the Lagrange multiplier on the resource constraint (62). Hence μ_0 is constant in time. After tax profits are redistributed lumpsum to latitude x consumers in the fraction $s_{FF}(x,t)$ and proceeds from taxes are redistributed lump sum to latitude x consumers in the fraction $s_{Tax}(x,t)$.

The FONC conditions for the fossil fuel firms are

$$p(t)(1-\theta(t)) = \mu_0 \exp(\int_{s=0}^{t} r(s)ds), \text{ or} \tag{63}$$

$$\left[\mathbb{A}\Omega F_q' - \tau(x,t)\right](1-\theta(t)) = \mu_0 \exp(\int_{s=0}^{t} r(s)ds) \tag{64}$$

3.4 Equilibrium

In any decentralized problem consumption goods firms at latitude x will choose demands $K(x,t)$ and $q(x,t)$ to set

$$r(t) + \delta = \mathbb{A}\Omega F_K', \ p(t) + \tau(x,t) = \mathbb{A}\Omega F_q' \tag{65}$$

Conditions (58) and (64), for a multiplier value $\bar{\mu}_0$ that exhaust the fossil fuels reserves, along with the optimality conditions for immobile labour will determine the equilibrium temporal and latitudinal paths for K and q. Since firms take temperature and taxes as parametric these paths can be written, denoting by (^{e}) equilibrium, as:[21]

$$\{C^e(x,t; D, \tau, \theta, p), K^e(x,t; T, \tau, \theta, p), q^e(x,t; T, \tau, \theta, p)\}_{x=-1}^{x=1} \tag{66}$$

---

[21] Since firms take taxes as parametric the paths will also depend on taxes.

4   Optimal Fossil Fuel Taxation

We consider a social planner the seeks to internalize the climate externality, by choosing paths for fossil fuel taxes or profit taxes on world fossil fuel firms in order to maximize the social welfare measure defined by (36) subject to the constraints for the evolution of the temperature and the carbon dioxide concentration, the global fossil fuel constraint and the resource constraint at each location or

$$C(x,t) + \dot{K}(x,t) + \delta K(x,t) = \mathbb{A}(x,t)\Omega(T(x,t))F(K(x,t), L(x,t), q(x,t)) \tag{67}$$

In order to provide a more clear picture of the impact of thermal diffusion on fossil fuel taxes we simplify climate dynamics following section 2.1 by assuming that the temperature dynamics are modelled by the two-mode approximation, that the co-albedo function is independent of the temperature field or $\alpha(x, T(x,t)) = \alpha(x)$, and that $\sigma(x) = \sigma$ and $\beta(t) = \beta$. From (8)-(12) it can be seen that the zero-mode depends on the concentration $M(t)$ but not on the thermal diffusion coefficient D, while the the second mode depends on D but not on $M(t)$.[22] Then from (8) the zero mode dynamics can be written as

$$\dot{T}_0 = -T_0 - \frac{A}{B} + \int_{x=-1}^{x=1} QS(x)\alpha(x)\,dx + \frac{2\sigma\xi}{B}\ln\left(\frac{M(t)}{M_0}\right) \tag{68}$$

$$\dot{T}_0 = -T_0 + Z_1 \ln\left(\frac{M(t)}{M_0}\right) + Z_0, \tag{69}$$

$$Z_1 = \frac{2\sigma\xi}{B}, Z_0 = -\frac{A}{B} + \int_{x=-1}^{x=1} QS(x)\alpha(x)\,dx \tag{70}$$

The temperature field can then be written as $T(x,t) = T_0(T) + T_2(t,D)P_2(x)$. The Hamiltonian for the planner is:

$$
\begin{aligned}
H \;=\; & \int_x \left\{ v(x)L(x,t)\left[U\left(\frac{C(x,t)}{L(x,t)}\right) - \Omega_C(T_0(t) + T_2(t,D)P_2(x))\right] \right. \\
& + \lambda_K(x,t)\left[\mathbb{A}(x,t)\Omega(T_0(T) + T_2(t,D)P_2(x))F(K(x,t), L(x,t), q(x,t))\right. \\
& \left.\left. - C(x,t) - \delta K(x,t)\right]\right\}\,dx + \lambda_M(t)\left[-mM(t) + \beta\int_x q(x,t)\,dx\right] \\
& + \lambda_{T_0}(t)\left[-T_0 + Z_1\ln\left(\frac{M(t)}{M_0}\right) + Z_0\right] + \mu_R\left[R_0 - \int_0^\infty\int_x q(x,t)\,dxdt\right]
\end{aligned} \tag{71}
$$

---

[22] This is because from (10) when $\sigma(x) = \sigma$ we have that $\int_{-1}^{1}\xi\sigma\ln\frac{M(t)}{M}P_2(x)\,dx = 0$ since $\int_{-1}^{1}P_2(x)\,dx = 0$.

The optimality conditions imply:

$$v(x) U'\left(\frac{C(x,t)}{L(x,t)}\right) = \lambda_K(x,t) \tag{72}$$

$$\lambda_K(x,t) \mathbb{A}\Omega F_q' + \beta \lambda_M(t) = \mu_R \tag{73}$$

$$\dot{\lambda}_K(x,t) = \left(\rho + \delta - \mathbb{A}\Omega F_k'\right) \lambda_K(x,t) \tag{74}$$

$$\dot{\lambda}_M(t) = (\rho + m) \lambda_M(t) - \lambda_{T_0}(t) \frac{Z_1}{M(t)} \tag{75}$$

$$\dot{\lambda}_{T_0}(t) = (\rho + 1) \lambda_{T_0}(t) + v(x) L(x,t)\Omega_{C,T_0}' - \lambda_K(x,t) \mathbb{A}\Omega_{T_0}' F \tag{76}$$

Using the market equilibrium conditions (52, 58, 59, 63) we obtain $v(x) \Lambda(x)e^{(\rho t - R(t))} = \lambda_K(x,t)$ and the optimal fossil fuel and profit taxes respectively as:

$$\tau^*(x,t) = \frac{\mu_R - \beta \lambda_M(t;D)}{v(x) \Lambda(x)e^{(\rho t - R(t))}} - p(t) = \frac{\mu_R - \beta \lambda_M(t;D)}{\lambda_K(x,t;D)} - p(t) \tag{77}$$

$$p(t) = \frac{\mu_0 e^{\Gamma(t)}}{1 - \theta^*(t)} \tag{78}$$

The dependence of the tax functions on the thermal diffusion coefficient follows from the fact that damage functions depend on D through their dependence on the temperature field which is given by $T(x,t) = T_0(T) + T_2(t,D) P_2(x)$. The climate externality is captured by the costate variable $\lambda_M(t;D)$. As we will show in the next section $\lambda_M^*(t;D) < 0$, therefore as expected, when we account for the climate externality fossil fuel taxes increase. From (77) and (78) the following result follows:

Proposition 2 *The optimal tax on fossil fuel is uniform across latitudes if* $v(x) \Lambda(x) = 1$. *If* $v(x) = 1$ *then the poorer latitudes should pay a relatively lower tax on fossil fuel use.*

From (77) it is clear that if $v(x) \Lambda(x) = 1$, that is the welfare weights are the inverse of the present value of the marginal utility at location x, then we see that taxes are independent of x. If $v(x) = 1$, then (77) implies for a latitude x and the temperate latitude 0 that

$$\frac{p(t) + \tau^*(x,t)}{p(t) + \tau^*(0,t)} = \frac{\Lambda(0)}{\Lambda(x)} \tag{79}$$

Since it is plausible to assume that $\Lambda(0) > \Lambda(x)$ because the Equator is poorer, it follows that $\tau^*(0,t) < \tau^*(x,t)$ for all t.

The case where $v(x) \Lambda(x) = 1$ can be regarded as the solution when the Negishi weights are used. Another way to obtain spatially uniform fossil fuel taxes is to assume that intertemporal endowment flows are adjusted so that $\Lambda(x) = \Lambda(x')$ and $\lambda_K(x,t) = \lambda_K(x)$ and set $v(x) = 1$. In both cases

the spatially uniform fossil fuel taxes will implement the global welfare optimum of section 3.2. Thus this proposition suggests that if international transfers across locations are not possible or the Negishi solution cannot be implemented then fossil fuel taxes should not be uniform across latitudes. This result was first noted by Chichilnisky and Heal (1994), while similar results have been obtained by Sandmo (2006), Anthoff (2011), Keen and Kostogiannis (2011). Our result which is obtained in the context of an one-dimensional EBCM, provide an explicit spatial structure for the fossil fuel taxes and imply that these taxes depend on the thermal diffusion coefficient D through the second mode. In fact it would be more accurate to write $\tau^* (x, t; D)$. An important policy issue is the size of bias introduced on optimal taxes when heat transportation is ignored. The bias can be defined as

$$|\tau^* (x, t; D) - \tau^* (x, t; D \to \infty)| \tag{80}$$

since when $D \to \infty$ the second and all higher modes vanish and it is that average global temperature and distribution of temperature across latitudes that determined damages.

4.1 The Temporal Profile of Optimal Taxes

As stated in the introduction one of the purposes of this paper was to provide insights regarding the optimal time profile for current and future mitigation. In terms of the model developed this means the study of the temporal profiles of spatially uniform optimal taxes on fossil fuels $\tau(t)$ and the profits tax $\theta(t)$ that implements a competitive equilibrium that is the same as the solution of the global social welfare problem. This means that we assume that international transfers have been implemented so that $\Lambda(x) = \Lambda(x')$ and $\lambda_K(x, t) = \lambda_K(x', t) = \lambda_K(t)$ for all x.

If we take the time derivative of (63) we obtain

$$\frac{d\left[p(t)(1 - \theta^*(t))\right]/dt}{p(t)(1 - \theta^*(t))} = r(t) = \mathbb{A}\Omega F_K' - \delta \tag{81}$$

which is the Hotelling's rule indicating that after tax marginal profits increase at the rate of interest.

Let us examine the cases of profits taxes and unit fossil fuel taxes separately. We examine profit taxes by setting $\tau(t) = 0$ and unit taxes by setting $\theta(t) = 0$ respectively. From (81) the optimal profit tax function should satisfy

$$-\frac{\dot{\theta}^*(t)}{1 - \theta^*(t)} = r(t) - \frac{\dot{p}(t)}{p(t)}, \tag{82}$$

while the optimal unit tax function should satisfy

$$\frac{(\dot{p}(t) - \dot{\tau}^*(t))}{(p(t) - \tau^*(t))} = r(t) \tag{83}$$

The policy ramp under the gradualist approach suggests that $\dot{\tau}^*(t) > 0, \dot{\theta}^*(t) > 0$. To examine the validity of this result in the context of our model, we seek to locate sufficient conditions so that profit tax and/or the unit tax will decline through time. In order to have a declining tax schedule through time equation (82) implies that (84) below must hold.

$$r(t) - \frac{\dot{p}(t)}{p(t)} > 0, \tag{84}$$

Note that a declining tax schedule through time contrast dramatically with the gradualist tax schedule which increases through time. Since we are implementing the global welfare optimum we use the optimality conditions of section 3.1 without the two-mode approximation of the temperature dynamics. We denote by (*) the global welfare optimizing paths.

Lemma 1 $\zeta(t) \equiv \int_x \sigma \lambda_T^*(t, x; D) dx < 0$, $\lambda_M^*(t; D) < 0$.

For the proof see Appendix 4. The lemma states an intuitive result. If we denote by V^* the maximum value function for the welfare maximization problem, we know from optimal control results that if V^* is differentiable, $\frac{\partial V^*}{\partial T(x,t)} = \lambda_T^*(x, t; D)$. That is $\lambda_T(x, t; D)$ can be interpreted as the shadow value of temperature at time t and latitude x. Thus $\zeta(t) \equiv \int_x \sigma \lambda_T(t, x; D) < 0$, for fixed σ, can be interpreted as the global shadow cost of temperature at time t across all latitudes, which means that an increase in temperature across all latitudes will reduce welfare. In a similar way $\lambda_M^*(t; D) < 0$ means that an increase in atmospheric accumulation of CO_2 at any time t will reduce welfare.

Proposition 3 *If $m < \delta$, then the optimal profit tax decreases through time, or $\dot{\theta}^*(t) < 0$. Furthermore, the optimal unit tax on fossil fuels grows at a rate less than the rate of interest, or $\frac{\dot{\tau}^*(t)}{\tau^*(t)} < r^*(t)$.*

Proof. Set $\tau(t) = 0$. For a decreasing $\theta(t)$ (84) should hold. At the global social welfare maximizing path, after omitting $(x, t; D)$ to ease notation, we have that

$$p^*(t) = (F_{total}^*)_q' = \frac{\mu_R^* - \beta\lambda_M^*}{\lambda_K^*} \tag{85}$$

$$r^* = \rho - \frac{\dot{\lambda}_K^*}{\lambda_K^*} \tag{86}$$

then

$$r^* - \frac{\dot{p}^*}{p^*} = \rho - \frac{\dot{\lambda}_K^*}{\lambda_K^*} - \frac{d[(\mu_R^* - \beta\lambda_M^*)/\lambda_K^*]/dt}{[(\mu_R^* - \beta\lambda_M^*)/\lambda_K^*]}, \tag{87}$$

or using the optimality conditions for the costate variables

$$r^* - \frac{\dot{p}^*}{p^*} = \tag{88}$$

$$\frac{\beta\lambda_M^*(m - \delta) - (\beta\xi/BM(t)) \int_X \sigma\lambda_T^* dx}{(\mu_R^* - \beta\lambda_M^*)}$$

To show that $\theta^*(t)$ decreases through time we need to show that the numerator and the denominator of (88) are both positive. If the decay of atmospheric carbon dioxide is slow so that $m - \delta < 0$, then by lemma 1 the numerator is positive. From the Kuhn-Tucker conditions μ_R^* is non-negative and $\lambda_M^* < 0$ by lemma 1. Therefore $\dot{\theta}^*(t) < 0$. To examine the time path of the optimal unit tax we use (83) to obtain

$$\dot{\tau}^*(t) = \dot{p}^*(t) - r^*(t)p^*(t) + r^*(t)\tau^*(t) \tag{89}$$

We want to locate sufficient conditions for $\dot{p}^*(t) - r^*(t)p^*(t) < 0$. But this is true if and only if $r^* - \dot{p}^*/p^* > 0$ which was our previous result. Therefore $\dot{\tau}^*(t)/\tau^*(t) < r^*(t)$. ∎

Thus, we have essentially produced sufficient conditions for rapid ramp-up of profit taxes and for unit taxes to rise at a rate less than the net of depreciation rate of return $r^*(t)$ on capital.[23]

Condition (88) provides also sufficient conditions for an increasing tax schedule according to gradualist approach.

Proposition 4 *If $m > \delta$ and $\lambda_M^*(m - \delta) - \left(\frac{\xi}{BM(t)}\right) \int_X \sigma\lambda_T^* dx > 0$ then $\dot{\theta}^*(t) > 0$ and $\frac{\dot{\tau}^*(t)}{\tau^*(t)} = \frac{\dot{p}^*(t) - r^*(t)p^*(t)}{\tau^*(t)} + r^*(t) > r^*(t)$.*

The first part of the proposition follows directly from lemma 1 and Proposition 2. For the second part note that when $\dot{\theta}^*(t) > 0$, $r^* - \dot{p}^*/p^* < 0$ and $\dot{p}^*(t) - r^*(t)p^*(t) > 0$. Thus a gradualist tax schedule requires, in the context of our model, rapid decay of the atmospheric carbon dioxide, and a relatively small global shadow cost of temperature at time t across all latitudes. In this case profit taxes on fossil fuel firms are increasing, and unit taxes on fossil fuels increase higher than the rate of interest.

4.1.1 The impact of co-albedo and thermal transportation

The welfare optimum, the market equilibrium and the optimal taxes were obtained by explicitly accounting for the co-albedo function and the thermal

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[23] For similar results on discrete time model with full depreciation of capital in one period see Golosov et al. (2011).

diffusion across latitudes. An emerging question is whether this complication in modeling is significant and whether we can trace, in a tractable way, the impact of thermal diffusion across latitudes. We provide two preliminary results here.

(i) **The discounting function effect.** As it is shown in the proof of Lemma 1 if we denoted by $\int_x \sigma \lambda_T (t,x)\, dx \equiv \zeta$ the global shadow cost of temperature across latitudes, then $\dot{\zeta} = \upsilon \zeta - \Xi(t)$, where

$$\upsilon \equiv \rho + 1 - \frac{Q}{B} \left(\frac{\int_x \sigma \lambda_T (x,t;D)\, S(x)\, (\partial a/\partial T)\, dx}{\int_x \sigma \lambda_T (x,t;D)\, dx} \right)$$

Thus the impact of T on co-albedo (since $\partial a/\partial T > 0$) causes the discounting function υ to fall which will make the forward discounted costs of climate change induced by burning an extra unit of fossil fuels higher than when the co-albedo function is independent of temperature, or $\partial a/\partial T = 0$. This could be very important quantitatively if the impact of T on $a(x, T(x,t))$ can vary by latitude as well as be quite large due to effects on types of plant growth and other determinants of co-albedo besides ice.

(ii) The damage effect. This effect relates to the impact of thermal transportation on damages across locations. To explore this impact we use (21), (22) and the two-mode approximation. Note first that if D goes to infinity, as we have shown in Proposition 1 only the mode zero remains and thermal transportation does not affect damages across latitudes. Hence if $\Omega(x, T(x,t))$ does not depend explicitly upon x but only on $T(x,t)$[24], then we have:

$$
\begin{aligned}
J(t; \infty) &= \int_x \left\{ \frac{[\mathbb{A}(x,0)L(x,0)^{\alpha_L}\Omega(T_0(t;\infty))]^{1/\alpha_L}}{\int_{x'} \mathbb{A}(x',0)^{1/\alpha_L}L(x',0)\Omega(T_0(t;\infty))^{1/\alpha_L}\,dx']^{\alpha_K+\alpha_q}} \right\} dx \\
&= \Omega(T_0(t;\infty)) \int_x \left\{ \frac{\int_x[\mathbb{A}(x,0)^{1/\alpha_L}L(x,0)]}{\int_{x'} \mathbb{A}(x',0)^{1/\alpha_L}L(x',0)dx]^{\alpha_K+\alpha_q}} \right\} dx \quad (90)
\end{aligned}
$$

because $1 = \alpha_L + \alpha_K + \alpha_q$ by the assumption of constant returns. When D is finite this convenient factoring will not take place because in the two mode case $\Omega(T_0(t;D) + T_2(t;D)P_2(x))$ even when damages to production do not depend upon x independently of $T(x,t)$. Therefore one measure of how much heat transport, as reflected by damages D, matters at date t is $|J(t;\infty) - J(t;D)|$.

5   Approximations and Numerical Simulations

In the previous section we derived general results regarding the optimal mitigation policies and their time profiles, and the relative importance of

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[24]This means that damages depend on location, only through the latitude dependent temperature function.

introducing heat transport into the model. In this section we try to use approximating solutions and in particular the two mode-solution along with some additional simplifications of the climate and the economic model, in order to provide analytically tractable results regarding the latitude dependent temperature and damage functions.

5.1 Simplifications of the Climate Model

We use the two-mode approximating solution with the simplifications: that the co-albedo function does not explicitly depend on $T(t)$ and can be written as $a(x) = a_0 - a_1 P_2(x)$; that $S(x) = 0.5[1 - s_0 P_2(x)]$, with ($a_0 = 0.681$, $a_1 = 0.202$, $s_0 = 0.477$) (North at al. 1981); and that $\sigma(x) = \sigma$ independent of x. The two-mode approximating ODEs become

$$\frac{dT_0}{dt} = -\frac{A}{B} - T_0(t) + \frac{1}{B}\left[\langle QS(x)\alpha(x), 1\rangle + \xi \ln\frac{M(t)}{M_0}\langle\sigma, 1\rangle\right] \quad (91)$$

$$\frac{dT_2}{dt} = -(1 + 6D)T_2(t) + \frac{5}{2B}\langle QS(x)\alpha(x), P_2(x)\rangle \quad (92)$$

Assume that T_0 and T_2 are evolving in faster time scale than M and that they relax fast to their respective steady states, so we assume $\frac{dT_0}{dt} = \frac{dT_2}{dt} = 0$. Then temperature can be expressed as a function of M as:

$$\hat{T}(x, t; D) = C_0 + C_1 \ln\frac{M(t)}{M_0} - \frac{C_2}{(1 + 6D)}P_2(x) \ , C_0, C_1, C_2 > 0 \quad (93)$$
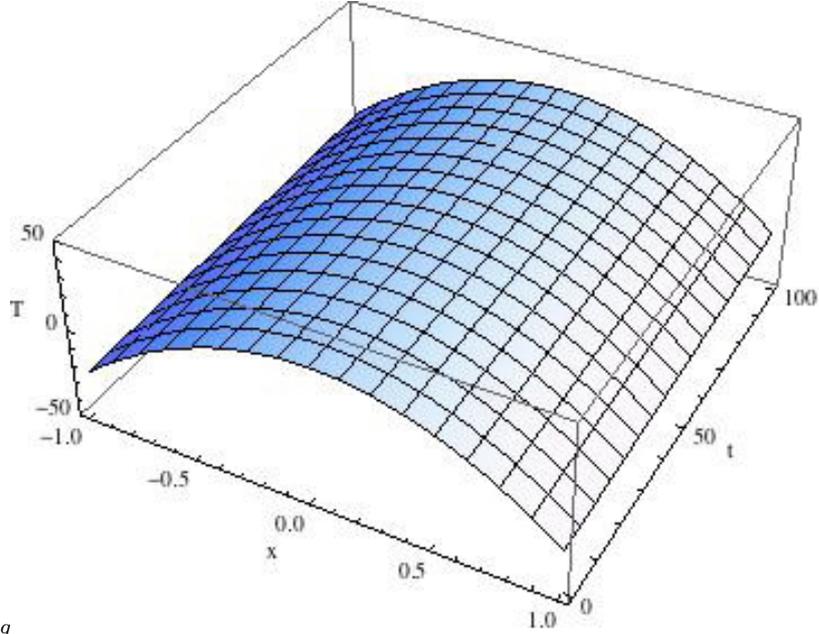
We approximate this temperature function, which is shown in figure 1 with $t = 0$ corresponding t0 year 2011. The parameterizations based on North and exogenous emissions growth at 1,026% per year according to the IPCC-A1F1 scenario was assumed.[25]

This temperature function implies a current average temperature of approximately 24°C for the equator and -25°C for the poles. It is worth noting that similar temperature functions have been derived by climate scientists (e.g. Sellers 1969, 1976), but without the impact of human activities on climate. In our case this impact is realized by the increase in the concentration of atmospheric carbon dioxide. When $D \to \infty$ the temperature function is spatially homogeneous or 'flat' across latitudes The distinction between a latitude dependent and a flat temperature field provides a first sign of the impact of thermal transport on the estimation of the temperature function.

To approximate damages for a given damage function we need to map changes in temperature measured from a benchmark period to numbers indicating the level of damages. Define the temperature anomaly as $T^+(x, t; D) = \hat{T}(x, t; D) - T_0(x, t)$, where $T_0(x, t)$ is the distribution of temperature across

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[25] We use $A = 203.4; B = 2.09; Q = 310; \sigma = 0.7; \xi = 5, 35$, $D = 0.649, M_0 = 583$GTC.

1

₁.jpg

Figure 1: A latitude dependent temperature function

latitudes as implied by existing data for a benchmark period. The temperature anomaly is shown in figure 2, with $t = 0$ corresponding to 2011.[26] Figure 2 indicates that and approximation of a spatial dependent temperature anomaly between the average of 1890-1900 and the current period suggests $0.8°C$ for the equator and $1.7°C$ for the Poles.

Then the damage function can be defined as function of the temperature anomaly $\hat{T}(x, t; D) - T_0(x, t)$ as $\Omega(T^+(x, t; D))$

To derive a damage function for the latitude dependent model we use a functional form for a damage function associated with production losses due to climate change, which is similar to the one used in the recent version of the DICE model (Nordhaus 2007c, Appendix B), or

$$\Omega\left(T^+(x, t; D)\right) = \frac{1}{1 + \omega(x)\,\theta_2\left(T^+(x, t; D)\right)^2}\quad \theta_2 = 0.0028388. \quad (94)$$

[26]To approximate the temperature anomaly $T^+(x, t; D)$ we used as benchmark the increase in temperature for eight zonal latitudes (90N-64N, 64N-44N, 44N-24N, 24N-Equator, Equator-24S, 24S-44S, 44S-64S, 64S-90S) which is reported by NASA, between the average of the years 1880-1900 and 1951-1980. The results indicate that the temperature has increased more at high altitudes relative to the equator. The data for calculating the change in the concentration of atmospheric CO_2, between average 1880-1900, 1950-1980, and 2001, which were used to calibrate the temperature field, were taken from the annual series of concentrations in ppm of the Mauna Loa data (ftp://ftp.cmdl.noaa.gov/ccg/co2/trends/co2_annmean_mlo.txt) and . Etheridge et al. (1998) ice cores data.
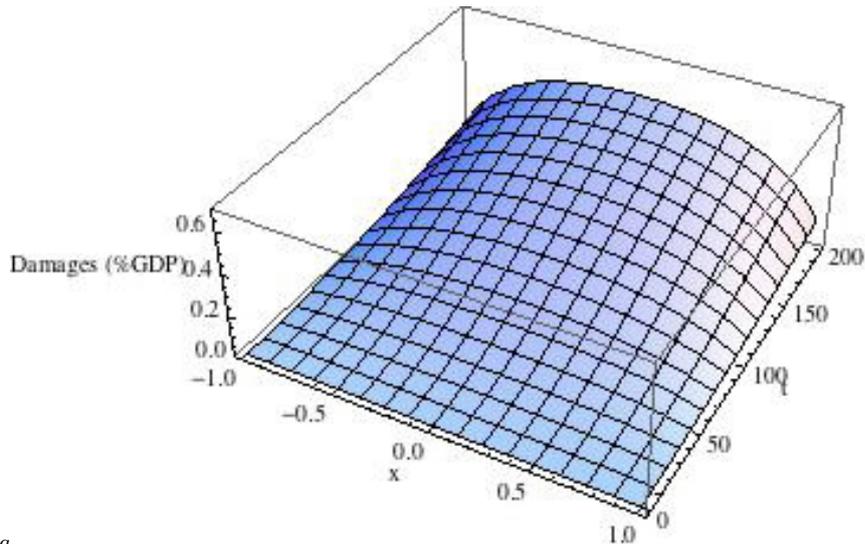
2
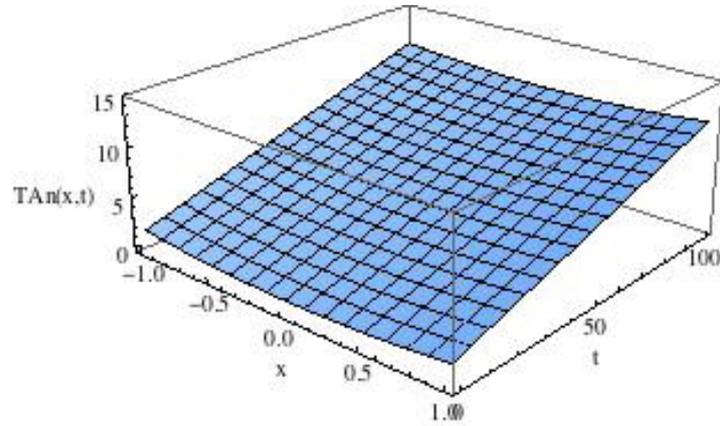
2.jpg

Figure 2: The temperature anomaly

Using this temperature anomaly, the resulting damage function is presented in figure 3 in the form $1 - \Omega$, that is damages are measured in terms of proportional losses in GDP across latitudes.[27]

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3.jpg

Figure 3: Latitude dependent damages

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[27] $\omega(x)$ is a concave function with a maximum at $x = 0$. This function reflects the generally accepted fact that damages as proportion is GDP are expected to be higher at low latitudes (Nordhaus and Boyer 1999, Chapter 4, Nordhaus 2007b). This latitude dependence of the parameters of the damage function, makes marginal damages from a given increase in temperature higher around the equator. $\omega(x)$ was calibrated in the following way. Let a damage function $g(x,t) = \left(1 - \alpha x^2\right) \theta_2 T^+ (t)^2$. Then relative marginal damages between the equator ($x = 0$) and a given lattitude y will be $\frac{1}{1 - \alpha y^2}$. In our simulation $\alpha = 0.85$, which implies that marginal damages around the equator zone will be twice the marginal damages around $50°$ North for the same change in temperature.

This damages function predicts relative high damages as proportion of GDP around the equator. The damage function described here can be also used to incorporate damage reservoirs, Any source of damages that acts like a "damage reservoir" where the reservoir can be exhausted and once it is exhausted there is no further damages emanating from that source can motivate a damage function capturing damage reservoirs. As we have seen above, polar ice or permafrost can be regarded as a damage reservoir, i.e. a "pool" of potential damages that's latitude dependent. The established relative higher temperature anomaly at high latitudes suggests the possible existence of such damage reservoirs. A damage function incorporating damage reservoirs should have two components. The first, approximating the damage reservoir, will show high damages initially which will decline with time as temperature increases and the reservoir is exhausted. As the reservoir is exhausted the second conventional component of the damage, e.g. the one shown in figure 4 will dominate. The use of existing scientific data to approximate a damage function incorporating damage reservoirs is an interesting area of further research, with potentially important policy implications.

5.2   Climate Response Functions

In the simplified climate model with an exogenous growth of the atmospheric CO_2 concentration, $M(t) = M_0 \exp(gt)$, we can define climate response functions (CRF). CRFs are functions that determine the changes in temperature and damages at latitude x and time t resulting from an exogenous changes of the atmospheric CO_2 concentration. To obtain a CRF, consider the steady state of the two-mode approximation ODEs in the more general case where the co-albedo function depends on temperature:

$$0 = -\frac{A}{B} - T_0(t) + \tag{95}$$
$$\frac{1}{B}\left[\langle QS(x)\alpha(T_0(t) + T_2(t)P_2(x)), 1\rangle + \xi \ln\frac{M(t)}{M_0}\langle\sigma(x), 1\rangle\right]$$
$$0 = -(1 + 6D)T_2(t; D) + \tag{96}$$
$$\frac{5}{2B}\left[\langle QS(x)\alpha(T_0(t) + T_2(t)P_2(x)), P_2(x)\rangle + \xi \ln\frac{M(t)}{M_0}\langle\sigma(x), P_2(x)\rangle\right]$$

A CRF can be defined in two stages. First for a change in $M(t)$ determine the change in $T_0(x, t)$ and $T_2(x, t)$, and then for the changes in T_0 and T_2 determine the change in $\Omega(T(x, t)) \equiv \Omega(T_0(t) + T_2(t)P_2(x))$. To determine the first stage changes the comparative static matrix of (95), (96) is

$$\begin{bmatrix} -1 + \frac{1}{B}\frac{\partial\langle QS\alpha, 1\rangle}{\partial T_0} & \frac{1}{B}\frac{\partial\langle QS\alpha, 1\rangle}{\partial T_2} \\ \frac{5}{2B}\frac{\partial\langle QS\alpha, P_2\rangle}{\partial T_2} & -6D\frac{\partial\langle QS\alpha, P_2\rangle}{\partial T_0} \end{bmatrix}\begin{bmatrix} \frac{dT_0}{dM} \\ \frac{dT_2}{dM} \end{bmatrix} = \begin{bmatrix} -\frac{\xi}{M(t)}\langle\sigma(x), 1\rangle \\ -\frac{5}{2B}\frac{\xi}{M(t)}\langle\sigma, P_2\rangle \end{bmatrix} \tag{97}$$

Assuming that the determinant Δ_c of the comparative static matrix is not zero and denote by $, \partial_{T_0,M}, \partial_{T_2,M}$ the comparative static derivatives the changes in T_0 and T_2 and temperature T are given by

$$dT_0(t) = (\partial_{T_0,M}) dM(t), \; dT_2(t) = (\partial_{T_2,M}) dM(t) \qquad (98)$$

$$dT(t,x) = dT_0(t) + P_2(x) dT_2(t) = \qquad (99)$$
$$[(\partial_{T_0,M}) + (\partial_{T_2,M}) P_2(x)] dM(t)$$

The impact on damages will then be determined as:

$$d\Omega(T(x,t)) = \Omega'_T [dT_0(t) + P_2(x) dT_2(t)] = \qquad (100)$$
$$\Omega'_T [(\partial_{T_0,M}) + (\partial_{T_2,M}) P_2(x)] dM(t)$$

In the simplified climate model (91), (92) and (93), $D_{T_0,M} = C_1/M(t)$ and $D_{T_2,M} = 0$ and

$$dT(t,x) = C_1 \frac{dM(t)}{M(t)}, d\Omega(x, M(t)) = \Omega'_T C_1 \frac{dM(t)}{M(t)}, \text{or} \qquad (101)$$

$$dT(t,x) = 5.11962 \frac{\Delta M(t)}{M(t)} \qquad (102)$$

using our parametrization. The CRFs can be constructed for both the utility related damages $\Omega_C(T(x,t))$ and the production related damages $\Omega_F(T(x,t))$. The new element that we provide relative to the existing literature (e.g. Mendelsohn 1999) is that our spatial models allows us to predict the spatial damages in specific locations and thus answer questions regarding the geographical distribution of the impacts from an increase in atmospheric carbon dioxide.

6   Concluding Remarks

In this paper we develop a model of climate change consisting of a one-dimensional energy balance climate model which is coupled with a model of economic growth. In our economy output is produced by capital, labour and a globally finite amount of fossil fuels. The use of fossil fuels in production generates emissions which accumulate in the atmosphere and block outgoing solar radiation, increasing thus the temperature. In our one-dimensional model heat diffuses across latitudes.

We believe that modeling heat diffusion in the coupled model is the main contribution of our paper since it allows, for first time as far as we know, the derivation of latitude dependent temperature, damage and climate response functions, as well as optimal mitigation policies which are all determined endogenously through the interaction of climate dynamics with optimizing forward looking economic agents.

Our results suggest that if the international transfers required to attain a globally Pareto optimal solution cannot be implemented, then taxes on fossil fuels should be lower in relatively poorer geographical zones. The degree of geographical tax differentiation depends on the heat diffusion across latitudes. The issue of whether international transfers of consumption goods to compensate for climate damages can be implemented is far from settled. If we consider a world with two goods, consumption goods and environmental services, where utility is modeled by a CES function, then the magnitude of the elasticity of substitution between consumption and environment is important regarding the possibility of compensations.[28] Even if consumption of material goods is growing exponentially, utility growth is bounded in the long run by the growth in climate services which in the long run are bounded and likely to be declining if global climate change is not controlled. If the elasticity of substitution between material consumption and environment is less than one, it will not be possible in the long run for wealthy latitudes to compensate damaged latitudes. Without appropriate implementation of international transfers taxes should be latitude specific and there sizes should depend on the heat transfer across locations.

We also provide results indicating that if the decay of atmospheric CO_2 is lower than the depreciation of capital then profit taxes on fossil fuel firms will decline over time and unit taxes on fossil fuels will grow at a rate less then the rate of interest. These results, which can be contrasted with the gradually increasing policy ramps derived by IAM models like DICE or RICE indicate that mitigation policies should be stronger now relative to the future. Increasing policy ramps so that mitigation is stronger in the future require rapid decay of the atmospheric carbon dioxide, and a relatively small global shadow cost of temperature increase. Since the decay of the atmospheric carbon dioxide is an empirical issue we hope that our analysis provides a basis for justifying the gradualist or the early mitigation approach.

Our model is a surface EBCM where the impact of oceans is reflected in the carbon decay parameter m, but not further modeling is undertaken. EBCMs can be augmented with a deep ocean component that redistributes vertically the heat energy via uniform vertical diffusion (Kim and North 1992). Our methodology can be easily extended to include the deep ocean component with vertical diffusion. This is another important task for future research.

The one-dimensional model allows the exploration of issues which cannot be fully analyzed in conventional zero-dimensional models. In particular one-dimensional models with spatially dependent co-albedo allow the introduction of latitude depended damage reservoirs like endogenous ice-lines and

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[28] For a detailed analysis of this argument in the economics of climate change see Sterner and Persson (2008), Heal (2009).

permafrost. We believe that if further research leads to a realistic parametrization of damage reservoirs or thresholds the type of coupled models discussed here could lead to sensible calibrations of damage function and policy ramps. Since reservoir damages are expected to arrive relatively early and diminish in the distant future - because the reservoir will be exhausted - the temporal profile of the policy ramp could be declining, enforcing the result obtained for profit taxes, or even U-shaped. A U-shaped policy ramp could explained by the fact that as high initial damages due to the reservoir will start declining as the reservoir is exhausted, giving rice to a declining policy ramp, damages from the increase of the overall temperature will dominate causing the policy ramp to become increasing. This is another potentially interesting and important area of further research.

Appendix 1: The two mode solution

In this appendix we show how to derive the two mode solution (8-(12). We start with the basic PDE with temperature as the state variable which is defined using (1) as:

$$B\frac{\partial T(x,t)}{\partial t} = QS(x)\alpha(x, T(x,t)) - [(A + BT(x,t)) - h(x,t)] + DB\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial T(x,t)}{\partial x}\right]$$

(103)

The two mode solution is defined as:

$$\hat{T}(x,t) = T_0(t) + T_2(t, D)P_2(x), \ \ P_2(x) = \frac{(3x^2 - 1)}{2}$$

(104)

then, after dropping D to ease notation

$$\frac{\partial T(x,t)}{\partial t} = \frac{dT_0(t)}{dt} + \frac{dT_2(t)}{dt}P_2(x)$$

(105)

$$\frac{\partial T(x,t)}{\partial x} = T_2(t)\frac{dP_2(x)}{dx} = T_2(t)3x$$

(106)

Substitute the above derivatives and using the definition of $h(x,t)$ into (103) to obtain:

$$B\frac{dT_0(t)}{dt} + B\frac{dT_2(t)}{dt}P_2(x) = QS(x)\alpha(x, x_s) -$$

(107)

$$\left[(A + B(T_0(t) + T_2(t)P_2(x))) - \sigma(x)\xi\ln\frac{M(t)}{M_0}\right] +$$

$$BD\frac{\partial}{\partial x}\left[(1-x^2)T_2(t)\frac{\partial P_2(x)}{\partial x}\right] \ , \text{or}$$

(108)

$$B\frac{dT_0(t)}{dt} + B\frac{dT_2(t)}{dt}P_2(x) = QS(x,t)\alpha(x, x_s) - A -$$

(109)

$$BT_0(t) - BT_2(t)P_2(x) + \sigma(x)\xi\ln\frac{M(t)}{M_0} - 6DBT_2(t)P_2(x)$$

Use:

$$\int_{-1}^{1} P_n(x)P_m(x)dx = \langle P_n(x), P_m(x)\rangle = \frac{2\delta_{nm}}{2n+1}$$

(110)

$$\delta_{nm} = 0 \text{ for } n \neq m, \delta_{nm} = 1 \text{ for } n = 1$$

and note that $P_0(x) = 1, \ P_2(x) = \frac{(3x^2-1)}{2}$

Multiply (109) by $P_0(x)$ and integrate from -1 to 1 to obtain

$$B\frac{dT_0(t)}{dt} + B\frac{dT_2(t)}{dt}\langle P_0(x), P_2(x)\rangle = \int_0^1 QS(x,t)\alpha(x,\hat{T}(x,t))P_0(x)dx - A$$

$$BT_0(t) - BT_2(t)\langle P_0(x), P_2(x)\rangle + \xi\ln\frac{M(t)}{M_0}\int_{-1}^1\sigma(x)\,dx-$$

$$6DBT_2(t)\langle P_0(x), P_2(x)\rangle \text{ , or}$$

$$B\frac{dT_0(t)}{dt} = -A - BT_0(t)+$$

$$\int_{-1}^1\left[QS(x,t)\alpha(x,x_s) + \xi\ln\frac{M(t)}{M_0}\sigma(x)\right]dx \qquad (111)$$

Multiply (109) by $P_2(x)$ and integrate from -1 to 1 noting that $\int_{-1}^1 P_2(x)dx = 0$, and $\langle P_2(x), P_2(x)\rangle = \frac{1}{5}$ to obtain

$$B\frac{dT_0(t)}{dt}\int_{-1}^1 P_2(x)dx + B\frac{dT_2(t)}{dt}\langle P_2(x), P_2(x)\rangle =$$

$$\int_{-1}^1 QS(x,t)\alpha(x,x_s)P_2(x)dx - A - BT_0(t)\int_{-1}^1 P_2(x)dx - BT_2(t)\langle P_2(x), P_2(x)\rangle +$$

$$\xi\ln\frac{M(t)}{M_0}\int_{-1}^1\sigma(x)P_2(x)dx - 6DBT_2(t)\langle P_2(x), P_2(x)\rangle \text{ , or}$$

$$\frac{2}{5}\frac{dT_2(t)}{dt} = \left[\int_{-1}^1 QS(x,t)\alpha(x,x_s) + \xi\ln\frac{M(t)}{M_0}\sigma(x)\right]P_2(x)dx-$$

$$\frac{2}{5}BT_2(t) - \frac{12}{5}DBT_2(t) \text{ , or}$$

$$B\frac{dT_2(t)}{dt} = -B(1+6D)T_2(t)+$$

$$\frac{5}{2}\left[\int_{-1}^1 QS(x,t)\alpha(x,x_s) + \xi\ln\frac{M(t)}{M_0}\sigma(x)\right]P_2(x)dx \qquad (112)$$

The ODEs (111) and (112) are the ODEs of the two mode solution. The solutions of these ODEs shown in follow from standard methods.□

Appendix 2: Proof of Proposition 1

Differential equation (10) can be written as $\dot{T}_2 = -(1+6D)T_2 + (5/2B)\,\Phi(t)$. As $D \to \infty$ any steady state of (10) defined as $T_2^+ = \frac{5\Phi(t)}{2B(1+6D)} \to 0$. Furthermore, consider the ODE

$$\frac{d\bar{T}_2}{dt} = -(1+6D)\bar{T}_2 + (5/2B)\,UB. \qquad (113)$$

Since $\dot{T}_2 \leq -(1+6D)T_2 + (5/2B)\,UB$, then by Gronwall's inequality the solution of (10) will be bounded above by the solution $\bar{T}_2(t)$ of (113). This solution however goes to zero as $D \to \infty$. Therefore $T_2(t) \to 0$ as $D \to \infty$.□

Appendix 3: Proof of (44)

The relevant part for the Maximum Principle derivation associated with the Hamiltonian (37) is

$$... + \int_X \lambda_T(t,x) \frac{1}{B} \left[QS(x)\alpha(x,T(x,t)) - \left[(A + BT(x,t)) - \sigma(x)\xi \ln \frac{M(t)}{M_0} \right] + DB\frac{\partial}{\partial x} \left[(1-x^2)\frac{\partial T(x,t)}{\partial x} \right] \right] dx \tag{114}$$

Put

$$v \equiv \frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial T(x,t)}{\partial x}\right], u \equiv \lambda_T(t,x) \tag{115}$$

Then

$$\int_X \lambda_T(t,x)\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial T(x,t)}{\partial x}\right]dx = \left. uv\right|_{x=-1}^{x=1} - \int_{x=-1}^{x=1} vdu =$$

$$-(1-x^2)\frac{\partial T(x,t)}{\partial x}\partial\lambda_T(t,x)\,dx = -\int_{x=-1}^{x=1}\left[(1-x^2)\partial\lambda_T(t,x)\right]\frac{\partial T(x,t)}{\partial x}dx$$

since the term

$$\left. uv\right|_{x=-1}^{x=1} = \left. \partial/\partial x[(1-x^2)\partial\lambda_T(x,t)/\partial x]dx\right|_{x=-1}^{x=1} = 0 \tag{116}$$

is zero because it is zero at $x = -1$ and $x = 1$.

Put

$$v \equiv T(x,t), u \equiv (1-x^2)\partial\lambda_T(t,x) \tag{117}$$

and integrate by parts once more to obtain:

$$-\int_{x=-1}^{x=1}\left[(1-x^2)\partial\lambda_T(t,x)\right]\frac{\partial T(x,t)}{\partial x}dx = \int_{x=-1}^{x=1}T(x,t)\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial\lambda_T(t,x)}{\partial x}\right]dx \tag{118}$$

If we take the partial derivative of the Hamiltonian (37) with respect to $T(x,t)$ for each (x,t), we will obtain **(44)**.□

Appendix 4: Proof of Lemma 1

The costate variables for the welfare maximization problem satisfy

$$\dot\lambda_M(t) = (\rho + m)\lambda_M(t) - \frac{\xi}{BM^*(t)}\int_{-1}^{1}\sigma(x)\lambda_T(t,x) \tag{119}$$

$$\dot\lambda_T(t,x) = (\rho+1)\lambda_T(t,x) + \frac{\upsilon(x)L(t,x)\partial\Omega_c(T(t,x))}{\partial T} - \lambda_K(t)\frac{\partial F_{to}}{\partial T} \tag{120}$$

$$\frac{\lambda_T(t,x)}{B}\frac{\partial QS(x)\alpha(x,T(x,t))}{\partial T} - D\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial\lambda_T(x,t)}{\partial x}\right]$$

We know that $\frac{\partial F_{to}}{\partial T} < 0$, since $\partial\Omega(x,T(x,t))/\partial T(x,t) < 0$ by assumption,

also by assumption $\partial\Omega_C(x, T(x,t))/\partial T(x,t) > 0$ and $\partial a(x, T(x,t))/\partial T(x,t) > 0$. To show that $\lambda_M^*(x,t) < 0$, It is enough to locate sufficient conditions for $\int_x \sigma(x)\lambda_T^*(x,t)dx < 0$. This is easy to do for the case $\sigma(x)$ constant in x. Multiply the costate equation for $T(x,t)$ by σ and integrate with respect to x to obtain:

$$\frac{d}{dt}(\int_x \sigma\lambda_T(t,x)\,dx) = \tag{121}$$

$$\left[\rho + 1 - \frac{Q}{B}\left(\frac{\int_x \sigma\lambda_T(t,x)S(x)a_T'dx}{\int_x \sigma\lambda_T(t,x)\,dx}\right)\right]\int_x \sigma\lambda_T(t,x)\,dx$$

$$+ \int_x \sigma\left(\upsilon(x)L\Omega_{C,T}' - \lambda_K F_{to,T}'\right)dx - \int_x \sigma\partial/\partial x[(1-x^2)\partial\lambda_T(x,t)/\partial x]dx$$

Note that the term

$$\int_x \sigma\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial\lambda_T(x,t)}{\partial x}\right]dx = 0 \tag{122}$$

is zero since it is an integral of a derivative of a term from $x = -1$ to $x = +1$ and that term is zero at $x = -1$ and $x = 1$. Put $\int_x \sigma\lambda_T(t,x)\,dx \equiv \zeta(t)$ and rewrite (121) as:

$$\dot{\zeta} = \left[\rho + 1 - \frac{Q}{B}\left(\frac{\int_x \sigma\lambda_T(x,t)S(x)a_T'}{\int_x \sigma\lambda_T(x,t)\,dx}dx\right)\right]\zeta \tag{123}$$

$$+ \int_x \sigma\left(\upsilon(x)L\Omega_{C,T}' - \lambda_K F_{to,T}'\right)dx \tag{124}$$

or as:

$$\dot{\zeta}(t) = \phi(t)\zeta(t) + \int_x \sigma\left[\upsilon(x)L\Omega_{C,T}' - \lambda_K F_{to,T}'\right]dx \tag{125}$$

where

$$\phi(t) \equiv \rho + 1 - \frac{Q}{B}\left(\frac{\int_x \sigma\lambda_T S(x)a_T'dx}{\int_x \sigma\lambda_T dx}\right) \tag{126}$$

is a time varying discount factor. Since $F_{to,T}' < 0, \Omega_{C,T}' > 0$ by assumption, we see that $\sigma\int_x \lambda_T(t,x)\,dx \equiv \zeta(t) < 0$ for all t by forward integration, since:

$$\zeta(t) = -\left(\exp\int_0^t \phi(s)\,ds\right)\int_0^t \left[\exp\left(-\int_0^t \phi(s)\,dt\right)Z(s)\right]ds \tag{127}$$

$$Z(t) = \int_x \sigma\left[\upsilon(x)L\Omega_{C,T}' - \lambda_K F_{to,T}'\right]dx \tag{128}$$

From (43)

$$\dot{\lambda}_M(t) = (\rho + m)\lambda_M(t) - \frac{\xi}{BM(t)}\zeta(t) \tag{129}$$

which implies

$$\lambda_M(t) = e^{(\rho+m)t}\int_0^t e^{-(\rho+m)s}\frac{\xi\zeta(s)}{BM(s)}ds$$

Thus solving (129) forward for each t shows that λ_M is a forward integral of negative quantities for each t, therefore $\lambda_M(x,t) < 0$ for each t.$\square$

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