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# Numerical computation of the optimal vector field in a fishery model

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# Abstract

Many of the optimal control models analyzed in economics are formulated as discounted infinite time horizon problems, where the occurring functions are nonlinear as well in the states as in the controls. As a consequence solutions can often only be found numerically. Moreover, the long run optimal solutions are in the overwhelming cases limit sets like equilibria and/or limit cycles. Using these "trivial" solutions a BVP approach together with a continuation technique is used to calculate the parameter dependent dynamic structure of the optimal vector field. We use a one-dimensional optimal control model of fishery to exemplify the numerical techniques. But these methods, as will be shown, are applicable to a much wider class of optimal control problems with any number of state and control variables.

*Keywords:* Optimal vector field, BVP, Continuation, Multiple optimal solutions, Threshold point

#### 1. Introduction

During the last decades optimal control models have successfully been applied in economics and/or ecology. Many of these models exhibit non-convexities and featuring multi system parameter values. As a consequence analytical methods alone are not sufficient for a full analysis. Nonlinearities and non-convexities often also give rise to the occurrence of multiple or history-dependent solutions. Therefore a well adapted numerical framework is of need, allowing an efficient handling of qualitatively different solutions.

The occurring phenomena are manifold. These range from history-dependence, i.e., dependent on the initial value one converges to different attractors, and/or multiplicity, where for specific points the decision maker is indifferent choosing between different optimal solutions, converging to distinct long run optimal solutions or even converging to the same attractor. This multitude of solution behavior, corresponding to non-uniqueness or existence of different long run attractors, is scattered over a wide variety of articles, where a few of them are mentioned below.

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The literature at this point lacks a theoretical and well structured description of these different phenomena, which correspond to structural changes in the optimal solution. Thus they can be further ascribed as bifurcations of the optimal vector field. Developing a comprehensive bifurcation theory of the optimal vector field is however done in Kiseleva and Wagener (2010) for the one-dimensional shallow lake model. But this type of general framework is still missing for two and higher dimensional systems.

Some remarks regarding the notation, classifying the phenomena of multiplicity and/or dependence on the initial values, should however be given. We already mentioned that in this context not only a theoretical framework is missing, but also a standardized notation is lacking. For points exhibiting multiple optimal solutions an often used term is that of a Skiba point, or as introduced by myself in Grass et al. (2008) DNSS point, recognizing the individual contributions of the authors Dechert, Nishimura, Skiba, and Sethi. But a general definition is still missing and the use of an acronym becomes more and more confusing since this phenomenon has already been described by authors years before that of Sethi or Skiba, see Clark (1976). Therefore we will dismiss our own terminology of DNSS points and instead adopt to the same terminology that was introduced in Kiseleva and Wagener (2010). Even though this terminology has also been criticized it turns out advantageous compared to unnecessarily long and complicated acronyms.

Since bifurcations of the optimal vector field often correspond to global bifurcations of the canonical system derived from the underlying optimal control problem, (see, e.g., Wagener, 2003, 2006), efficient numerical tools are necessary for the calculations. Moreover the use of high-level illustrations and animations are also of great help for the development of a bifurcation theory, since these pictures provide the necessary intuition for the facts which then have to be proved rigorously. The aim of this article is a well adapted numerical approach derived from Pontryagin's Maximum Principle yielding a BVP which is combined with a continuation technique. A preliminary version of these numerical tools are implemented as a MATLAB package OCMat, which can be downloaded at http://orcos.tuwien.ac.at/research/ocmat\_software.

To exemplify these numerical techniques we use a one state, one control fishery model. This model is a simplified version of a three state optimal control problem, formulated in Crépin (2007). But these numerical techniques can in principle be applied to models with an arbitrary number of states, controls and constraints. In fact the MATLAB package OCMat has already been successfully applied to a number of different models, e.g., (Caulkins et al., 2005a, 2007, 2005b, 2008, 2010, 2009; Zeiler et al., 2010; Levy et al., 2006).

Subsequently we present a numerical technique for analyzing discounted problems over an infinite time horizon, referred to as (DIP), of the following

$$\max_{u(\cdot)} \int_0^\infty e^{-rt} g(x(t), u(t)) dt \tag{1}$$

s.t. 
$$\dot{x}(t) = f(x(t), u(t)), \quad t \in [0, \infty)$$
 (2)

$$ith \quad x(0) = x_0 \tag{3}$$

$$c(x(t), u(t)) \ge 0, \quad t \in [0, \infty).$$
 (4)

with  $x(t) \in \mathbb{R}^n$ ,  $u(t) \in \mathbb{R}^m$  and the state dynamics  $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ , the objective function  $g : \mathbb{R}^{n+m} \to \mathbb{R}$ , and the mixed path constraints  $c : \mathbb{R}^{n+m} \to \mathbb{R}^l$ , are assumed to be two times continuously differentiable in their arguments.

W

Next we formulate assumptions which have to be proved analytically for a concrete model, or if an exact proof is not possible one has to be aware that the numerical results only provides extremals, i.e., candidates for optimal solutions.

- **Assumption 1.** 1. There exists an optimal solution  $(x^*(\cdot), u^*(\cdot))$  for every  $x_0 \in C \subset \mathbb{R}^n$ , with C a compact set.
  - 2. Every optimal solution  $(x^*(\cdot), u^*(\cdot))$  converges to an equilibrium.
  - 3. The constraints (4) satisfy the constraint qualification along an optimal solution.

Next we state assumptions which can be made less restrictive

- **Assumption 2.** 1. The control variable u appears nonlinearly either in g(x, u) or f(x, u).
  - 2. The optimal control values can be stated as explicit functions of the state and costate variables, i.e.,  $u^*(t) = u(x(t), \lambda(t))$ . For active mixed constraints the Lagrangian multipliers can explicitly stated as functions of the state and costate variables, i.e.,  $\psi(t) = \psi(x(t), \lambda(t))$ .

**Definition 1.** Let  $(x^*(\cdot), u^*(\cdot))$  be a solution of model (DIP), then its limit set is called a *long run optimal solution*, and an optimal solution with  $(x^*(\cdot), u^*(\cdot)) \equiv (\hat{x}, \hat{u})$  is called an *equilibrium solution*.

# 2. A fishery model

The full fishery model in a coral reef presented in Crépin (2007) consists, at time t, of the three states fish (x(t)), algae (a(t)), coral (c(t)), and the control (u(t)) representing the effort of fishing. In our simplified version we will set the number of algae and coral constant, i.e.  $a(\cdot) \equiv a$  and  $c(\cdot) \equiv c$ , yielding a one state, one control optimal control problem.

The dynamics of fish is described by a logistic growth term (G), with a carrying capacity depending linearly on the number of algae, yielding

$$G(t) := x(t) \left( 1 - \frac{x(t)}{a} \right).$$

type

Additionally we have a term stemming from predation (P) which is decreasing with the number of corals, because of coral giving shelter to the fish. This term can be described by

$$P(t) := \frac{1}{(c+\tau)} \frac{x(t)^2}{1+x(t)^2}.$$

The stock of fishes is also reduced by fishing (u(t)x(t)), yielding the total fish dynamics as

$$\begin{split} \dot{x}(t) &= G(t) - P(t) - u(t)x(t) \\ &= x(t) \left(1 - \frac{x(t)}{a}\right) - \frac{1}{(c+\tau)} \frac{x(t)^2}{1 + x(t)^2} - u(t)x(t). \end{split}$$

To formulate the economic part of the model we simply assume that fishing yields positive gains from selling them at price p but also generates negative (ecological) side effects. Thus we define the total utility as

$$U(t) := pu(t)x(t) - u(t)^2$$

Summing up the optimal control problem, which is subsequently referred to as fishery model (FM), can be written as

$$\max_{u(\cdot)} \left\{ \int_0^\infty e^{-rt} \left( pu(t)x(t) - u(t)^2 \right) dt \right\}$$
(5)

s.t. 
$$\dot{x}(t) = x(t)\left(1 - \frac{x(t)}{a}\right) - \frac{1}{(c+\tau)}\frac{x(t)^2}{1 + x(t)^2} - u(t)x(t)$$
 (6)

$$u(t) \ge 0$$
, for all  $t$  (7)

$$x(0) = x_0 > 0. (8)$$

For the analysis of this model Pontryagin's Maximum Principle is used, where the details are carried out in the next section.

# 3. Necessary optimality conditions

Defining the Hamiltonian and Lagrangian (augmented Hamiltonian) of problem (FM) as

$$\mathcal{H}(x,u,\lambda,\lambda_0) := \lambda_0 \left( pux - u^2 \right) + \lambda \left( x \left( 1 - \frac{x}{a} \right) - \frac{1}{(c+\tau)} \frac{x^2}{1+x^2} - ux \right) \tag{9}$$

$$\mathcal{L}(x, u, \lambda, \psi, \lambda_0) := \mathcal{H}(x, u, \lambda, \lambda_0) + \psi u \tag{10}$$

an optimal solution  $(x^*(\cdot), u^*(\cdot))$  of (FM) has to satisfy the following necessary optimality conditions. There exists a (piecewise) continuous function  $\psi(\cdot)$  and a (piecewise) differentiable function  $\lambda(\cdot)$  with  $(\lambda(t), \lambda_0) \neq (0, 0), t \geq 0$ , such that at every time point t, where  $u^*(\cdot)$  is continuous

$$u^*(t) \in \operatorname*{argmax}_{u \ge 0} \mathcal{H}(x^*(t), u(t), \lambda(t), \lambda_0)$$
(11)

 $\dot{\lambda}(t) = r\lambda(t) - \frac{\partial \mathcal{L}}{\partial x}(x^*(t), u^*(t), \lambda(t), \psi(t), \lambda_0)$ (12)

with the transversality condition

$$\lim_{t \to \infty} e^{-rt} \lambda(t) = 0.$$
(13)

In Appendix A it is proved that the problem (FM) is normal and therefore  $\lambda_0$  can be set to one and is subsequently omitted. Since the Hamiltonian (9) is strictly concave with respect to u the Hamiltonian maximizing condition (11) can be reformulated as

$$\frac{\partial \mathcal{L}}{\partial u}(x^*(t), u^*(t), \lambda(t), \psi(t)) = 0$$
(14a)

$$\psi(t) \ge 0 \tag{14b}$$

$$\psi(t)u^*(t) = 0. \tag{14c}$$

Moreover the strict concavity with respect to u implies the uniqueness of the optimal control value. As a result  $u^*(\cdot)$  is continuous and  $\lambda(\cdot)$  is continuously differentiable. Since the control constraint satisfies the constraint qualification the Lagrangian multiplier  $\psi(\cdot)$  is also continuous.

Analyzing (14a) in detail we find that the maximizer

$$u^{\circ} = \operatorname*{argmax}_{u \ge 0} \mathcal{H}(x, u, \lambda)$$

can be derived from

$$\frac{\partial \mathcal{H}}{\partial u} = px - 2u - \lambda x = 0 \Leftrightarrow u^{\bullet} = \frac{x}{2}(p - \lambda)$$
$$\frac{\partial \mathcal{L}}{\partial u} = px - 2u - \lambda x + \psi = 0$$

yielding

$$u^{\circ} = \begin{cases} u^{\bullet} & \text{for } u^{\bullet} > 0\\ 0 & \text{for } u^{\bullet} \le 0 \end{cases}$$
(15)

$$\psi = \begin{cases} 0 & \text{for } u^{\bullet} > 0 \\ -x(p-\lambda) & \text{for } u^{\bullet} \le 0. \end{cases}$$
(16)

Summing up an optimal solution  $(x^*(\cdot), u^*(\cdot))$  for the problem (FM) has to be searched among the solutions  $(x(\cdot), u(\cdot))$  of the so called canonical system given by

$$\dot{x}(t) = x(t)\left(1 - \frac{x(t)}{a}\right) - \frac{1}{(c+\tau)}\frac{x(t)^2}{1 + x(t)^2} - u^{\circ}(t)x(t)$$
(17a)

$$\dot{\lambda}(t) = \lambda(t) \left( r - 1 + \frac{2x(t)}{a} + \frac{2x(t)}{(c+\tau)(1+x(t)^2)^2} \right) - u^{\circ}(t)$$
(17b)

and

satisfying the boundary conditions

$$x(0) = x_0$$
 and  $\lim_{t \to \infty} e^{-rt} \lambda(t) = 0.$ 

From a numerical point of view the transversality condition (13) does not provide information to explicitly calculate solutions of the system (17a) and (17b). But paths  $(x(\cdot), \lambda(\cdot))$  converging to an equilibrium  $(\hat{x}, \hat{\lambda})$  trivially satisfy the transversality condition (13). For the model (FM) it is proved in Appendix B that the stable paths are already all candidates for an optimal solution.

Thus we restated the problem of finding candidates for optimal paths, into the problem of finding the equilibria  $(\hat{x}, \hat{\lambda})$  of the ODEs (17a) and (17b), and calculating the corresponding stable paths. In case of multiple equilibria and multiple solutions we have to compare the corresponding value of the objective function and choose the maximizer.

*Practical remark.* It may not always be possible to prove the (global) optimality of the numerically calculated solutions. In these cases one can try to prove local optimality using second order necessary optimality conditions. Otherwise one has to keep in mind that in general, without further analysis, only extremals, i.e., candidates for an optimal solution are identified.

#### 4. Calculating stable paths

To keep the notation compact and to avoid an overloaded usage of indices we set

$$y = \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} := \begin{pmatrix} x \\ \lambda \end{pmatrix}$$
, and  $\mu := (r, p, c, a, \tau)$ 

Moreover we remind the reader that the canonical system includes the maximized control value  $u^{\circ}(t)$  and in the constrained case the Lagrangian multiplier  $\psi(t)$ , where using Assumption 2, both of them can be expressed as terms of state(s) and costate(s), i.e.,

 $u^{\circ}(t) = u^{\circ}(x(t), \lambda(t))$  and  $\psi(t) = \psi(x(t), \lambda(t)).$ 

Substituting these expressions into the canonical system we can shortly write

$$\dot{y}(t) = g(y(t), \mu)$$

or even shorter

$$\dot{y}(t) = g(y(t)),\tag{18}$$

whenever the parameter values  $\mu$  can be assumed as constant.

Next we assume that for a specific set of parameter values  $\mu$  the system (18) exhibits a saddle  $\hat{y}$ . Thus the two eigenvalues  $\xi_{1,2}$  of the Jacobian matrix  $J(\hat{y})$  exhibit opposite signs, where we assume  $\xi_1 < 0 < \xi_2$ .

For a given initial state  $y_0^1$  we then have to find a solution path  $y(\cdot)$  of (18) satisfying

$$y^1(0) = y_0^1 \tag{19}$$

$$\lim_{t \to \infty} y(t) = \hat{y}.$$
(20)

In the terminology of Appendix C this determines a (two-point) boundary value problem (TBVP), with  $T = \infty$ . To solve this problem numerically we have to reformulate the asymptotic condition (20), translating the convergence property into some "finite" setting.

Different approaches are possible where we present the simplest one. By definition (see (C.5)) the solution path  $y(\cdot)$  satisfying (20) lies in the stable manifold of  $\hat{y}$  and reaches therefore the local stable manifold in some finite time.<sup>1</sup> The stable eigenspace of  $\hat{y}$  is given by

$$\mathbf{E}^{s}(\hat{y}) = \kappa \nu \in \mathbb{R}^{2}, \ \kappa \in \mathbb{R}$$
(21)

with  $\nu$  being the eigenvector corresponding to the negative eigenvalue  $\xi_1$ .

The manifold theorem (see Appendix C Th. 1) states that the stable eigenspace (21) is tangent to the local stable manifold at  $\hat{y}$ . Therefore the condition (20) can approximately be replaced by

$$(y(T) - \hat{y}) \in \mathcal{E}^s(\hat{y}),\tag{22}$$

for T large enough. Moreover the condition (22) is equivalent to the equation

$$F'(y(T) - \hat{y}) = 0$$
, with  $F := \nu^{\perp}$ . (23)

In the general case the matrix F is given by a basis spanning the orthogonal complement to the stable eigenspace. Its actual computation is described in Th. 3 of Appendix C.

#### 4.1. The boundary value problem

Summing up the problem of finding a stable path satisfying (20) can (approximatively) be calculated by solving (23).<sup>2</sup> Additionally we normalize the time interval [0, T] to [0, 1], which is helpful in case of different arcs with firstly unknown switching times (see, e.g., Sect. 5). Therefore a solution  $y(\cdot)$  starting at  $y_0^1$  and converging to  $\hat{y}$  can numerically be computed by fixing end time T (large enough) and solving the BVP

$$\dot{y}(t) = Tg(y(t)), \ t \in [0, 1]$$
(24a)

$$y^1(0) = y_0^1$$
 (24b)

$$F'(y(1) - \hat{y}) = 0, \tag{24c}$$

where (24c) is called the *asymptotic transversality condition*. The appearance of T in (24a) is due to the normalization of the time interval.

and

 $<sup>^{1}</sup>$ The reader should make herself aware that the stable manifold and paths on the stable manifold are two distinct objects. For the two-dimensional case this fact is hidden, since the stable manifold and the solution path coincide in the phase portrait.

 $<sup>^{2}</sup>$ To compute an unstable path the same approach can be used by reversing time.

Practical remark. The advantage of truncating the time interval to a fixed value is the linearity of the boundary condition (24c). But the apparent draw back is the freedom in choosing the truncation time T. Thus one has to take care that during the computations the chosen time T is large enough, i.e, the distance  $||y(T) - \hat{y}||$  keeps sufficiently small. If necessary the end time T could be adapted using continuation. Alternatively the end time T could be considered as a free parameter. Then a further equation has to be added, e.g.,  $||y(T) - \hat{y}|| = \varepsilon$ , with  $\varepsilon$  fixed.

*Practical remark.* Another possibility is a transformation of the time interval  $[0, \infty)$  to the interval [0, 1] using, e.g.

$$s = \frac{1}{1+t}$$

For a different transformation see, e.g., Kitzhofer et al. (2009). But in all these approaches, where the infinite time interval is transformed into a finite time interval, a singularity is introduced and therefore the used BVP solver needs to handle singular BVPs.

#### 4.2. Initializing the boundary value problem

Next we have to answer the question of how to solve the BVP (24) explicitly? Every BVP algorithm implemented in a specific solver needs as an input argument an initial (approximative) solution. Depending on the problem this initial solution can be a rather rough approximation, or has to be very close to the sought for solution. Anyhow, continuation techniques allow us to continue a solution once we have found at least one solution. Thus we search for a solution we can easily get. But the simplest possible solution of the BVP (24) is the equilibrium itself, provided we change the initial condition (24b) into

$$y^1(0) = \hat{y}^1. \tag{25}$$

For this BVP the trivial path  $y(\cdot) \equiv \hat{y}$  determines a solution and continuing this trivial solution lets us find the solution we actually want to compute. This strategy is theoretically founded, e.g., in Winkler (1985) or Krauskopf et al. (2007).

After having formulated the main framework, for computing paths converging to an equilibrium, we now apply this method to problems derived from our optimal control problem. Optimal control theory provides us then the different boundary conditions we have to consider, when analyzing paths switching between active and inactive constraints, different paths starting at the same initial state and converging to different equilibria, and so forth. These specifications will be treated in the subsequent sections.

# 5. Case of a unique equilibrium

Returning to the interpretation of (18) as the canonical system of the optimal control problem (FM) we have to distinguish between two cases, namely the case

of the control value lying in the interior of the control region, i.e., c(y) > 0, and the case of the control value lying at the boundary, i.e., c(y) = 0, (cf. (15)). To differentiate between these two specifications of the canonical system, we introduce the variables z and h for the (co)state variables and the dynamics of the boundary system.

Thus the two representations of the canonical system are denoted as the interior system

$$\dot{y}(t) = g(y(t)) \tag{26}$$

and the boundary system

$$\dot{z}(t) = h(z(t))$$
 with  $c(z(t)) = 0,$  (27)

satisfying the (mixed) constraint (4).

Let us assume that interior system (26) exhibits a (unique) saddle  $\hat{y}$ , where  $\nu$  is the eigenvector corresponding to the negative eigenvalue, and we want to find the stable path starting at the initial value  $y^1(0) = \zeta$ . To validate that the solution path satisfies the constraint (4) we define a tolerance  $\varepsilon > 0$  and check that a path  $y(\cdot)$  satisfies

$$c(y(t)) \ge \varepsilon$$
, for all  $t$ . (28)

Using the results from the previous section the following BVP has to be solved

$$\dot{y}(t) = Tg(y(t)), \ t \in [0, 1]$$
 (29a)

$$y^1(0) = \zeta \tag{29b}$$

$$F'(y(1) - \hat{y}) = 0, \ F := \nu^{\perp}.$$
 (29c)

We use a simple continuation technique with a fixed step width

$$\Delta := \frac{\zeta - \hat{y}^1}{n}, \ n \in \mathbb{N}$$

and continue the solution of (29), with changing initial condition (see Fig. 1)

$$y^{1}(0) = \zeta_{j} := \hat{y}^{1} + j\Delta, \ j = 0, \dots, n.$$

The solution path at step j is denoted as

$$y^{(j)}(\cdot).$$

During the continuation process (28) is checked. If at step k + 1 the constraint (28) is violated (for reasons of simplicity we assume that the violation occurs at the initial point, i.e.,  $c(y^{(j)}(0)) < \varepsilon$ ), the step width is reduced and the k-th solution is continued until it is violated again. Repeating this procedure we finally find at some step N an (admissible) solution

$$y^{(N)}(\cdot)$$
, with  $c(y^{(N)}(0)) \in [0,\varepsilon)$ ,

lying near the boundary, where the control constraint becomes active.

To calculate the solution further on we have to distinguish between two arcs of the solution path. The first arc with the control value at the boundary and the second arc with the control value lying in the interior of the control region. Thus we can set up an extended BVP with additional boundary conditions at the switching point. Denoting the unknown switching time as  $T_s$  we state this BVP as

$$\dot{z}(t) = T_s h(z(t)), \ t \in [0, \ 1]$$
(30a)

$$\dot{y}(t) = (T - T_s)g(y(t)), \ t \in [0, 1]$$
(30b)

z(1) = y(0)(30c)

$$c(y(0)) = 0 \tag{30d}$$
$$z^{1}(0) = \zeta \tag{30e}$$

(30e)

$$F'(y(1) - \hat{y}) = 0, \ F := \nu^{\perp}.$$
 (30f)

Note that we doubled the dimension of the ODEs to handle the switching between two arcs.<sup>3</sup> The new condition (30c) reflects the continuity of the state and costate at the switching point and (30d) is needed to determine the unknown switching time  $T_s$ .<sup>4</sup>

*Remark* 1 (Switching from the boundary to the interior control region). For an arc with the control value lying at the boundary of the control region the Lagrangian multiplier has to be positive. Therefore this condition has to be checked during continuation. In case that this condition is violated the same procedure yields an extended BVP, where condition (30d) is then replaced by the condition<sup>5</sup>

$$\psi(z(1)) = 0. \tag{31}$$

# 5.1. Initialization of (30)

To start the continuation for the extended system (30) we use the solution path  $y^{(N)}(\cdot)$  by adding the trivial arc

$$z(t) = y^{(N)}(0), \quad t \in [0, 1] \text{ and } T_s = 0.$$

This solution trivially satisfy the BVP (30) and can therefore be used as an initial solution for the subsequent continuation steps (see Fig. 2).

<sup>&</sup>lt;sup>3</sup>This is the usual procedure of transforming a multi-point BVP into a two-point BVP. For a boundary value solver being able to handle multi-point problems this step has to be adapted in an obvious way.

<sup>&</sup>lt;sup>4</sup>In cases of a discontinuous control this condition has to be replaced by the continuity of the Hamiltonian at the switching point.

<sup>&</sup>lt;sup>5</sup>In cases, where the Lagrangian multiplier is discontinuous at the switching point this condition has to be replaced by the continuity condition of the Hamiltonian.

Figure 1 (Animation): The left figure depicts the continuation process in the state-costate space, whereas the right figure shows it in the state-control space. Starting at the "trivial" solution, i.e. the equilibrium solution a continuation process using BVP (29) is used to compute the solution with initial condition  $y_0^1 = 4$ . For the actual computation the parameter values are set to a = 5 and c = 0.5 and the end time is fixed at T = 500.

Figure 2 (Animation): The left figure depicts the continuation process in the state-costate space, whereas the right figure shows it in the state-control space. Starting at the "trivial" solution, i.e., the equilibrium solution a continuation process using BVP (29) is used to compute the solution with initial condition  $y_0^1 = 0.01$ . During this process a violation of the control constraint is detected and therefore the BVP (30) is used for the further continuation steps. For the actual computation the parameter values are set to a = 5 and c = 0.5 and the end time is fixed at T = 500.

# 6. Multiple equilibria

In many optimal control problems of type (DIP) the canonical system exhibits multiple equilibria. Therefore another basic issue is the identification of the optimal solution among different extremals. In case that multiple optimal solutions exist, the main problem is to locate the so called indifference point(s). These are points in the state space at which different paths yield the same objective value, i.e., each of these paths are equally optimal. For a detailed discussion of such points see Grass et al. (2008) and Kiseleva and Wagener (2010).

*Practical remark.* In general the main focus is on saddles exhibiting a stable manifold of the same dimension as the number of states. The reason is simply that only for these saddles one can expect that the projection of the stable manifold covers (the interesting region of) the state space. In the one-dimensional case every saddle already satisfies this condition.

#### 6.1. One superior solution

Let us assume that the canonical system (18) exhibits two saddles  $\hat{y}_1$  and  $\hat{y}_2$  (with one-dimensional stable manifold). To determine if these equilibria are both long run optimal solutions or if one is superior to the other we have to compare the objective value of the stable paths converging to  $\hat{y}_1$  and  $\hat{y}_2$ . Thus we use the continuation algorithm presented in the previous section trying to determine the paths with initial value  $y^1(0) = \hat{y}_1^1$  and converging to  $\hat{y}_2$  and vice versa for  $y^1(0) = \hat{y}_2^1$  converging to  $\hat{y}_1$ .<sup>6</sup> An example of such a case, where one of the stable paths can be continued to the state value of the other equilibrium, is depicted in Fig. 3.

*Practical remark.* The continuation process may fail for different reasons. First of all there may exist an unstable node in between and one or both stable paths emanate from this equilibrium. In that case the BVP solver may fail to find a solution or the returned solution (for fixed T) returns a solution with the end point not near the equilibrium.

Secondly there may exist an intermediate unstable focus and one or both stable paths may spiral out. In this case the fixed step continuation fails to find a solution and aborts if the stable path starts bending back. This can be avoided by using an adaptive continuation method. But since in the one-dimensional case the back bended segment cannot be optimal at all it is not necessary to follow this back bending (see Fig. 3).

# 6.2. Indifference point

Next we analyze the case where none of the stable paths can be continued to the other equilibrium state but parts of the paths overlap in the projection to the state space. In these cases we expect the existence of an indifference point

<sup>&</sup>lt;sup>6</sup>In fact we can already stop the computation if one stable path can be continued to the other equilibrium state. Since in that case the continued solution is superior to the equilibrium solution at this point (cf. Grass et al., 2008, Prop. 3.23).

Figure 3 (Animation): In the left figure the control is depicted during the continuation process and the corresponding objective value evaluated by the Hamiltonian is shown in the right figure. First the stable path is continued from the left equilibrium in direction to the right equilibrium. The spiraling behavior of this path is followed by using the adaptive continuation algorithm. Next the stable path converging to the high equilibrium is computed. During the continuation process two switching points are detected. Finally it can be seen that the solution converging to the high equilibrium is superior and therefore the unique long run optimal solution.

 $x_I$ , where it is equally optimal to converge either to  $\hat{y}_1$  or  $\hat{y}_2$  (see Fig. 4). The decisive property is the equality of the objective value for the two solutions

$$y_1(\cdot), y_2(\cdot), \text{ with } y_1^1 = y_2^1 = x_I, \ y_1^2 \neq y_2^2.$$

Since the objective value is given by the Hamiltonian (divided by the discount rate) evaluated at the two (different) points  $y_1(0)$  and  $y_2(0)$ , see, e.g., Michel (1982), the last equality can be stated as a boundary condition

$$\mathcal{H}(y_1(0)) = \mathcal{H}(y_2(0)).$$

These properties allow us to formulate the BVP characterizing an indifference point. Let  $\nu_1$  and  $\nu_2$  be the eigenvectors corresponding to the negative eigenvalues of the saddles  $\hat{y}_1$  and  $\hat{y}_2$  and let us choose the two maybe different fixed truncation times  $T_1$  and  $T_2$  then the solutions starting at an indifference point are characterized by the following BVP

$$\dot{y}_1(t) = T_1 f_1(y_1(t)), \ t \in [0, 1]$$
 (32a)

$$\dot{y}_2(t) = T_2 f_2(y_2(t)), \ t \in [0, 1]$$
 (32b)

$$\begin{aligned} y_2(t) &= 1_{2,1}(y_2(t)), \ t \in [0, 1] \end{aligned} \tag{32b} \\ y_1^1(0) &= y_2^1(0) \end{aligned} \tag{32c} \\ \mathcal{H}(y_1(0)) &= \mathcal{H}(y_2(0)) \end{aligned}$$

$$\mathcal{H}(y_1(0)) = \mathcal{H}(y_2(0)) \tag{32d}$$

$$F'_1(y_1(1) - \hat{y}_1) = 0, \ F_1 := \nu_1^{\perp}$$
 (32e)

$$F'_2(y_2(1) - \hat{y}_2) = 0, \ F_2 := \nu_2^{\perp}.$$
 (32f)

The two ODEs (32a) and (32b) denote the two different solutions.<sup>7</sup> The condition (32c) states that both solutions have to start from the same initial state, namely the sought for indifference point  $x_I$  and (32d) formulates the equity in their objective value. Finally (32e) and (32f) are the usual asymptotic transversality conditions for the two solutions converging to the (different) equilibria  $\hat{y}_1$ and  $\hat{y}_2$ . It is easy to check that the problem is well formulated, in the sense that the number of equations equals the number of unknowns.

To determine an initial solution for the BVP (32) we can use the results from the previous continuations, where we roughly determine the intersection point of the Hamiltonian evaluated along the two paths. Then the solutions starting at this approximated intersection point are calculated and can then be used as an initial solution for (32).

*Remark* 2. We note that the path(s)  $y_1(\cdot)$  and/or  $y_2(\cdot)$  may be compounded of different arcs (active/inactive constraints) which is not explicitly stated for reasons of notational simplicity.

 $<sup>^{7}</sup>$ Note that the ODEs (32a) and (32b) can consist of different arcs exhibiting active and inactive control constraints.

Figure 4 (Animation): This animation depicts the steps of locating an indifference point. In the left figure the control is shown and in the right figure the corresponding Hamiltonian value. In the first step the stable paths of both equilibria are continued in direction of the other equilibrium state. since both paths are back bending the continuation is interrupted. The Hamiltonian intersect and the intersection point is approximatively calculated (gray line). Then the stable paths are computed starting at initial value given by this approximative value. Finally using this approximative solution the BVP (32) is solved yielding its exact location (red line).

#### 6.3. Threshold point

Beside the superiority of one of the equilibria or the existence of a threshold point we may encounter that an unstable node  $\hat{z}$  lies between two saddles  $\hat{y}_i$ , i =1, 2 and additionally it is a long run optimal solution, provided the initial state is the equilibrium state  $y(0) = \hat{z}^1$ . A necessary condition for this case is that the stable paths do not overlap in the state space. Theory tells us that the stable paths are tangential to the weak eigenvector  $\nu_w$ , i.e., the eigenvector corresponding to the lowest eigenvalue, near the unstable node. Thus we split up each stable path into a (stable) part converging to the corresponding saddle and an unstable part converging (for reversed time) to the unstable node and being tangential to the weak eigenvector.

To set up the corresponding BVP we denote the stable paths as  $y_i(\cdot)$ , i = 1, 2 and two unstable paths corresponding to the unstable node as  $z_i(\cdot)$ , i = 1, 2. The unstable paths are assumed to be tangential to the weak eigenvector. Moreover we fix the truncation times  $T_1$  and  $T_2$  choose some  $\varepsilon > 0$  and introduce the free time parameter values T and S. Then the BVP can be written as

$$\dot{y}_1(t) = T_1 f(y_1(t))), \mu), \ t \in [0, 1]$$
 (33a)

$$\dot{z}_1(t) = -Tf(z_1(t))), \mu), \ t \in [0, 1]$$
(33b)

$$\dot{y}_2(t) = T_2 g(y_2(t)), \mu), \ t \in [0, 1]$$
 (33c)

$$k_2(t) = -Sg(z_2(t)), \mu), \ t \in [0, 1]$$
(33d)

$$y_1(0) = z_1(0)$$
 (33e)

$$y_2(0) = z_2(0) \tag{331}$$

$$F_1(y_1(1) - y_1) = 0, \ F_1 := \nu_1^+ \tag{33g}$$

$$F_2(y_2(1) - y_2) = 0, \ F_2 := \nu_2^-$$
 (33h)

$$z_1(1) = z + s\varepsilon\nu_w \tag{331}$$

$$z_2(1) = \hat{z} - s\varepsilon\nu_w. \tag{33j}$$

The Equations (33a-33d) denote the dynamics for the stable and unstable paths, (33e) and (33f) assure hat the unstable and stable parts are continuously connected and the next two equations denote the usual asymptotic transversality conditions. The last two equations (33i) and (33j) are also asymptotic boundary conditions, assuring that the solution ends up at the linearized stable manifold, but since we have to guarantee that they do not overlap in the state space we also have to specify the specific branch of the weak unstable manifold. Note that the times T and S have to be handled as free parameter values, since the end points  $z_1(1)$  and  $z_2(1)$  are now fixed. The parameter  $s \in \{-1, 1\}$  has to be chosen such that the unstable paths have no overlap.

*Practical remark.* For practical usage the BVP (33) is rather bulky and to check if an unstable node is a threshold point or if there exists an indifference point it is more convenient to try to continue the stable paths to the unstable node. The fixed truncation time should be chosen large enough. If the continuation succeeds the unstable node cannot be a threshold point. But if continuation fails one has to check the exact reason. This cumbersome formulation of a rather simple issue is a result of the (locally) asymptotic unstability of the equilibrium  $\hat{z}$ . This property has the effect that every path in its neighborhood converges (in reversed time) to the equilibrium. Therefore the linear asymptotic transversality condition (33h) does not guarantee that the path ends up at the "correct" branch of the unstable eigenspace. Anyhow, since for higher dimensions the analogous case is represented by a saddle exhibiting a (one-)dimensional stable manifold the simpler criterion, with fixed end time(s), can be used.

#### 7. Bifurcations of the optimal vector field

In the previous sections we presented specifications of the BVPs for the cases that the optimal vector field exhibits unique or multiple solutions, but is "stable" to small disturbances of the parameter values. Next we analyze in more detail the numerical setting to determine the crossing between two qualitatively different optimal vector fields in the parameter space. Thus we calculate the bifurcations of the optimal vector field. In Kiseleva and Wagener (2010) the authors give a fairly general classification of possible codimension one and two bifurcations of an optimal vector field for a one-state optimal control problem of the type we are analyzing. Using these results we show how to set up the boundary conditions characterizing the different types of bifurcations.

#### 7.1. Indifference-attractor bifurcation (IAB)

In Wagener (2003) it has been proved that a heteroclinic bifurcation, a stable and unstable path of two saddles coincide (see Fig. 5), can give rise to the emergence of indifference points. And in Kiseleva and Wagener (2010) this bifurcation was classified as one of the possible codimension one bifurcations an optimal vector field can undergo. In Fig. 6 the typical situation before, at and after an indifference-attractor bifurcation is depicted.

To set up the BVP characterizing an heteroclinic connection we note that this means that the stable path  $y_1(\cdot, \mu_b)$  of an equilibrium  $\hat{y}_1(\mu_b)$  coincides with an unstable path  $y_2(\cdot, \mu_b)$  of an equilibrium  $\hat{y}_2(\mu_b)$ . For notational simplicity the parameter value  $\mu \in \mathbb{R}$  is assumed as a single value. Reversing time an unstable path becomes a stable path and we therefore find

$$\dot{y}_1(t,\mu) = T_1 f_1(y_1(t,\mu)), \mu), \ t \in [0,\ 1]$$
(34a)

$$\dot{y}_2(t,\mu) = -T_2 f_2(y_2(t,\mu),\mu), \ t \in [0,\ 1]$$
(34b)

$$y_1^1(0,\mu) = y_2^1(0,\mu) \tag{34c}$$

$$\psi(y_1(\cdot,\mu), y_2(\cdot,\mu)) = 0 \tag{34d}$$

$$F_1(\mu)'(y_1(1,\mu) - \hat{y}_1(\mu)) = 0, \ F_1(\mu) := \nu_1(\mu)^{\perp}$$
 (34e)

$$F_2(\mu)'(y_2(1,\mu) - \hat{y}_2(\mu)) = 0, \ F_2(\mu) := \nu_2(\mu)^{\perp}.$$
 (34f)

The truncation times  $T_1$  and  $T_2$  are once more assumed as fixed. Since (34c) is satisfied for every point on the (un)stable path the (phase) condition, (34d),

has to be provided which allows to choose one specific solution. The simplest possibility for such an equation is to fix, e.g., the first coordinate of the initial state to some value  $\xi$ , yielding

$$\psi(y_1(\cdot,\mu), y_2(\cdot,\mu)) := y_1^1(0,\mu) - \xi.$$

At this place  $\nu_2(\mu)$  denotes the eigenvector corresponding to the positive eigenvalue of the saddle  $\hat{y}_2$ .

Practical remark. The computation of the bifurcation parameter  $\mu_b$  is numerically more extensive than for the previous algorithms. The reason is simply that we have to determine a specific parameter value and therefore the parameter value changes during its computation. But a change of the parameter value induces a change of the equilibira and the corresponding linearized stable manifold.

Therefore we have to tackle the problem of the parameter value dependence. If the boundary value solver is able to handle differential-algebraic equations (DAEs) the BVP (34) can be extended by adding the algebraic equations

$$f_1(\hat{y}_1(\mu), \mu) = 0$$
  
$$f_2(\hat{y}_2(\mu), \mu) = 0,$$

and externally determine the asymptotic boundary condition vectors (matrices)  $F_1(\mu)$  and  $F_2(\mu)$  using some algorithm for the calculation of eigenvectors.

For the actual computations we used an even simpler approach, where the equilibria and asymptotic boundary condition matrices were externally computed using a solver for nonlinear equations and a solver for the computation of eigenvectors.

In a two dimensional parameter space we can in general find an IAB curve. Thus choosing two parameter variables  $(\mu_1, \mu_2)$  a continuation algorithm can be used to calculate this curve.

#### 7.2. Indifference-repeller bifurcation (IRB)

In Kiseleva and Wagener (2010) two types of indifference-repeller bifurcations are distinguished, denoted as bifurcation of type one (IRB1) and type two (IRB2). The latter occurs, when a threshold point turns into an unstable focus, i.e., the imaginary parts of the eigenvalues become non-zero, and therefore an indifference point appears. Such a bifurcation can simply be detected by considering the local properties of the Jacobian matrix evaluated at the threshold point (unstable node). It has to be checked by the algorithm in Sect. 6.2 if an indifference point appears at the same time.

An indifference-repeller bifurcation of type one occurs in the transition from a threshold to an indifference point and the eigenvalues of the unstable node remain real numbers. For that case it has been shown in Kiseleva and Wagener (2010) that at the bifurcation the strong unstable path of the unstable node coincides with the stable path of the saddle (see Fig. 7). To locate the exact parameter value of the bifurcation we can therefore set up the same BVP as for

Figure 5 (Animation): This figure depicts the process of finding an indifferenceattractor bifurcation for the parameter value a. Therefore the unstable path (dashed line) of the left equilibrium and the stable path (solid line) of the right equilibrium are continued to an initial state x(0) = 2 for a = 7.25. This is an approximation for the exact value a, where the heteroclinic connection occurs. Solving the BVP (34) determines the bifurcation parameter value as  $a_b = 7.3501$ and returns the heteroclinic connection between the two saddles (blue dots).

Figure 6 (Animation): The case of an indifference-attractor bifurcation is depicted for changing the parameter value a. The gray dot denotes a non-optimal equilibrium with part of its unstable manifold (dashed line). Decreasing a the stable path of the optimal equilibrium comes closer to the unstable path until they coincide in a heteroclinic connection. This characterizes the indifference attractor bifurcation which lets emerge an indifference point shown by the dashed black line.

Figure 7 (Animation): This figure depicts the locating of an indifference-repeller bifurcation for the parameter value a. Therefore the strong unstable path (dashed line) of the unstable node and the stable path (solid line) of the higher saddle are continued to an initial state x(0) = 1.26. Solving the BVP (34) (with adapted eigenvector  $\nu_2(\mu)$ ) determines the bifurcation parameter value as  $a_b = 5.0911$ , where the strong unstable path and the stable path are connected. The red line points into the direction of the strong unstable eigenvector.

the IAB, where  $\nu_2(\mu)$  of (34f) is now taken as the strong unstable eigenvector, i.e., corresponding to the largest eigenvalue.

*Practical remark.* For the calculation of the strong unstable path the fixing of the truncation time becomes more severe since the linear equation (34f) is not only satisfied by the orthogonality of the vector but also if the end point of the path and the equilibrium (nearly) coincide. Since in the neighborhood of an unstable equilibrium every path converges to this equilibrium (by reversed time) for a large truncation time (34f) could (numerically) be satisfied for a path near enough to the equilibrium but not lying on the strong unstable eigenspace.

## 7.3. Codimension two bifurcations (ISN) and (DIR)

More involved bifurcations are those of codimension two, which usually appear as a result of the coincidence of two bifurcations of codimension one. Once

more we refer the reader to Kiseleva and Wagener (2010), where the indifferencesaddle-node bifurcation (ISN) is defined as the coincidence of the two codimension one bifurcations IAB and IRB1.

# 7.3.1. ISN bifurcation

Thus for  $\mu_b \in \mathbb{R}^2$  there exist two equilibria, a saddle  $\hat{y}_1(\mu_b)$  and a saddlenode equilibrium  $\hat{y}_2(\mu_b)$ , with a heteroclinic connection path, given by the stable  $y_1(\cdot, \mu_b)$  and center manifold path  $y_2(\cdot, \mu_b)$ . Denoting the Jacobian matrix of  $\hat{y}_2(\mu)$  as  $J_2(\mu)$  the corresponding BVP can be stated as

$$\dot{y}_1(t,\mu) = T_1 f_1(y_1(t,\mu)), \mu), \ t \in [0,\ 1]$$
(35a)

$$\dot{y}_2(t,\mu) = -T_2 f_2(y_2(t,\mu),\mu), \ t \in [0,\ 1]$$
(35b)

(35c)

 $y_1^1(0,\mu) = y_2^1(0,\mu)$ 

$$\psi(y_1(\cdot,\mu), y_2(\cdot,\mu)) = 0 \tag{35d}$$

$$F_1(\mu)'(y_1(1,\mu) - \hat{y}_1(\mu)) = 0, \ F_1(\mu) := \nu_1(\mu)^{\perp}$$
(35e)

$$F_2(\mu)'(y_2(1,\mu) - \hat{y}_2(\mu)) = 0, \ F_2(\mu) := \nu_2(\mu)^{\perp}$$
(35f)

$$\det(J_2(\hat{y}_2(\mu), \mu)) = 0.$$
(35g)

The boundary conditions (35a–35f) are already known from the IAB (34), where only  $\nu_2(\mu)$  is the eigenvector corresponding to the eigenvalue zero. The last condition (35g) is the necessary condition for a saddle-node bifurcation.

# 7.3.2. DIR bifurcation

The second codimension two bifurcation is the double-indifference-repeller bifurcation (DIR). This bifurcation is characterized by two saddles and a degenerate node, where the stable path of one of the saddles coincides with the unstable path corresponding to the unique eigenvector of the degenerate node. Additionally there is no overlap of the stable manifolds (in the neighborhood of the degenerate node). To guarantee this last requirement we have to restrain to the cumbersome formulation of problem (33). Denoting the stable paths as  $y_i(\cdot, \mu)$ , the unstable paths as  $z_i(\cdot, \mu)$ , i = 1, 2 and the Jacobian matrix of the degenerate node as  $J_2(\mu)$  with the corresponding eigenvector  $\nu_d(\mu)$  the BVP for DIR becomes

$$\dot{y}_1(t,\mu) = T_1 f(y_1(t,\mu),\mu), \ t \in [0,\ 1]$$
 (36a)

$$\dot{z}_1(t,\mu) = -Tf(z_1(t,\mu),\mu), \ t \in [0,\ 1]$$
 (36b)

$$\dot{y}_2(t,\mu) = T_2 g(y_2(t,\mu),\mu), \ t \in [0,\ 1]$$
 (36c)

$$\dot{z}_2(t,\mu) = -Sg(z_2(t,\mu),\mu), \ t \in [0,\ 1]$$
(36d)

$$y_1(0,\mu) = z_1(0,\mu)$$
(36e)  
$$y_1(0,\mu) = z_1(0,\mu)$$
(36f)

$$y_2(0,\mu) = z_2(0,\mu)$$
(301)  
$$E'(x,(1,\mu)) = \hat{x}_1(\mu) = 0 \quad E(x,\mu) = (x)^{\perp}$$
(26a)

$$F_1(y_1(1,\mu) - y_1(\mu)) = 0, \ F_1 := \nu_1(\mu)^- \tag{30g}$$

$$F_2'(y_2(1,\mu) - \hat{y}_2(\mu)) = 0, \ F_2 := \nu_2(\mu)^{\perp}$$
(36h)

$$z_1(1,\mu) = \hat{z}(\mu) + s\varepsilon\nu_d(\mu) \tag{36i}$$

$$z_2(1,\mu) = \hat{z}(\mu) - s\varepsilon\nu_d(\mu) \tag{36j}$$

$$tr(J_2(\mu)) - 4 \det(J_2(\mu)) = 0.$$
(36k)

The conditions (36a-36j) are already described for the problem of a threshold point (33) and the last condition (36k) characterizes a degenerate node, i.e., an equilibrium in the transformation from an unstable focus to an unstable node.

*Practical remark.* For the actual computation the parameter values  $\mu$  cannot be varied totally free in a neighborhood of the bifurcation parameter, but have to be restricted to the (one-dimensional) manifold given by

$$\det(J_2(\mu)) = 0$$

or

$$tr(J_2(\mu)) - 4\det(J_2(\mu)) = 0$$

respectively. Thus an implemented algorithm has to assure that these requests are satisfied.

#### 8. Conclusion

We presented an algorithm for the numerical analysis of the optimal vector field as it usually occurs in the context of economic and ecological applications. These are discounted, infinite time horizon, autonomous and highly nonlinear problems, with only a few number of state and control variables. None of these restrictions are principal limitations of this algorithm, and in fact the author has used this algorithm already for a variety of models differing from the here presented class of models. Anyhow the number of state and control variables can be a severe restriction especially in the presence of various control constraints.

The main idea underlying the here presented method is quite simple. First the initial data of the problem are changed in a way, such that the solution (or more exact a solution candidate) can easily be computed. In our example these are the steady state solutions, thus the initial state  $y(0) = x_0$  is changed into  $y(0) = \hat{y}$ . Having computed such a "primitive" solution a pathfollowing algorithm is used to continue it to a solution we are actually interested. If this continuation process is carried out carefully possible occurring bifurcations deliver insights into the optimal behavior of the system. Therefore even if the continuation to the original data fails we gained important information about the solution structure. And of course the application of continuation is not restricted to the initial data  $x_0$  or parameter values, it can also be used to solve finite time horizon problems, non-autonomous problems, etc. Thus a further direction for future research work is an extension of this method to other types of models, e.g., differential games, delayed systems, etc.

The approach to use a boundary value solver together with a continuation technique let this algorithm be well adapted for the application to bifurcation problems of the optimal vector field. In future work the interplay between the boundary value solver and the pathfollowing procedure has to be implemented in a more sophisticated way to provide an efficient numerical tool for bifurcation analysis of the optimized system, like it exist already for usual nonlinear dynamics, e.g., with MatCont. That such an implementation can be realized, even in the case, where the boundary value solver is assumed as a "black box", has already been shown in Winkler (1985). Thus an immediate application of the here presented methods is the extension to higher dimensional systems and their numerical analysis, which will then include the occurrence of indifference and threshold manifolds, of limit cycles and the detection of their bifurcations.

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# Appendix A. Proving that (FM) is normal

Let us assume that  $(x^*(\cdot), u^*(\cdot))$  is an optimal solution of the problem (FM) and that  $\lambda_0 = 0$ . then the Hamiltonian maximizing condition (11) reduces to

$$\max_{u \ge 0} \lambda(t) \left( x(t) \left( 1 - \frac{x(t)}{a} \right) - \frac{1}{(c+\tau)} \frac{x(t)^2}{1 + x(t)^2} - ux(t) \right).$$
(A.1)

We note that u only appears linearly in (A.1) yielding that the maximum depends on the sign of

$$\frac{\partial \mathcal{H}}{\partial u} = -\lambda(t)x(t). \tag{A.2}$$

Since x(t) > 0 and  $\lambda(t) \ge p > 0$  has to be satisfied by the non-negativity of Lagrangian multiplier, it follows that  $u^*(t) = 0$ . From the dynamics (6) we therefore find that  $x^*(\cdot)$  converges to a stable equilibrium  $\hat{x} > 0$ . Considering

the utility function we see that in the neighborhood  $B(\hat{x},\varepsilon)$  of we can choose a positive value of u on a finite time span  $\Delta T$  such that

$$u(t)(px(t) - u(t)) > 0, \quad t \in [\tilde{T}, \tilde{T} + \Delta T]$$

yielding a positive objective value and therefore violating the optimality of  $u^*(t) = 0$ . This finally proves that  $\lambda_0$  can be set to one.

# Appendix B. Existence and convergence of an optimal solution

In this section we prove the assumptions Assumption 1 and Assumption 2 stated at the beginning. We start with the behavior of the solution for the time going to infinity. Considering the total time derivative of the optimal control value in the interior of the control region (see (15)), i.e,

$$\dot{u}(t) = \frac{\mathrm{d}}{\mathrm{d}t} \left( \frac{x(t)}{2} (p - \lambda(t)) \right)$$

Substituting the costate  $\lambda$  and the expressions for the dynamics finally yields

$$\dot{u}(t) = u(t)x(t)\left(\frac{1}{a} + \frac{1 - x(t)^2}{(c + \tau)(1 - x(t)^2)^2) + \frac{1}{2}}\right) - u(t)^2 + ru(t) - \frac{x(t)}{2}p\left(r - 1 + \frac{2x(t)}{a} + \frac{2x(t)}{(c + \tau)(1 - x(t)^2)}\right).$$
 (B.1)

On the boundary of the control region the dynamics is trivially zero.

Summing up we find from the state dynamics (6) and control dynamics (B.1) together with its extension on the boundary that there exist  $x_m > 0$  and  $u_m > 0$  such that the compact set  $[0, x_m] \times [0, u_m]$  is invariant under the dynamics. Since limit cycles cannot occur for a positive discount rate r > 0, see e.g., Wagener (2003), the Poincaré-Bendixson theorem assures that the paths converge to an equilibrium in the interior  $\hat{x} > 0$  and  $\hat{u} > 0$  or converge to an equilibrium at the boundary of the control region, i.e.,  $\hat{x} > 0$  and  $\hat{u} = 0$ . But repeating the arguments for the previous section we see that the latter case does not yield an optimal solution. Thus every optimal solution converges to an interior equilibrium. By the compactness of the region  $[0, x_m] \times [0, u_m]$  the existence of an optimal solution is assured, which finally proved that Assumption 1 does hold. The constraint qualification is trivially satisfied.

# Appendix C. Definitions and theorems for ODEs

Let  $f: \mathbb{R}^n \to \mathbb{R}^n$  and  $b: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$  then a problem specified by

$$\dot{x}(t) = f(x(t)), \ t \in [0, T]$$
 (C.1)

$$b(x(0), x(T)) = 0$$
 (C.2)

is called a (two-point) boundary value problem (TBVP or shortly BVP), and b(x(0), x(T)) = 0 are called the (two-point) boundary conditions. If (C.2) is replaced by a function  $b : \mathbb{R}^n \times \ldots \times \mathbb{R}^n \to \mathbb{R}^n$  with

$$b(x(0), x(t_1), \dots, x(t_m), x(T)), \quad 0 < t_1 < \dots < t_m < T$$
 (C.3)

problem (C.1) and (C.3) is called a multi-point boundary value problem (MBVP). Let  $\hat{x}$  be an equilibrium of the ODE (C.1) and  $\xi_i$ ,  $i = 1, \ldots, n$  be the eigenvalues of the Jacobian matrix  $J(\hat{x})$ . Then

$$\mathbf{E}^s = \operatorname{span}\{\nu_j \in \mathbb{R}^n : (\xi_j) < 0\}, \quad \dim \mathbf{E}^s = n_-$$
(C.4)

is called the stable eigenspace.

Let  $\hat{x}$  be an equilibrium of the ODE (C.1) and U be a neighborhood of  $\hat{x}$ ; then the set

$$\mathbf{W}^{s}_{loc}(\hat{x}) = \{ x \in U: \ x(0) = x, \ \lim_{t \to \infty} x(t) = \hat{x}, \ \text{and} \ x(t) \in U, t \geq 0 \}$$

is called the *stable local manifold* of  $\hat{x}$ . The set

$$W^{s}(\hat{x}) = \bigcup_{t \le 0} \{ x(t) : \ x(0) \in W^{s}_{loc}(\hat{x}) \}$$
(C.5)

is called the *global stable manifold* of  $\hat{x}$ . Reversing time gives you the definition for the unstable case.

Now the following theorem can be proved:

**Theorem 1** (Stable manifold theorem). Suppose that  $\hat{x}$  is an equilibrium of (C.1), where  $f \in C^k(\mathbb{R}^n)$ . Let  $n_-$  be the corresponding dimensions of the stable subspaces  $E^s$ . Then there locally exist the local stable  $C^k$  manifold  $W^s_{loc}(\hat{x})$  of the dimensions  $n_-$  being tangent to  $E^s, E^u$ .

A (hyperbolic) equilibrium  $\hat{x}$  satisfying  $0 < n_{-} < n$  is called a *saddle point* and a path converging to the saddle is called a *stable path* or *saddle path*.

**Theorem 2** (Characterization of the linearized stable manifold). Let  $\hat{x}$  be a saddle of (C.1). Then  $(x - \hat{x}) \in E^s(\hat{x})$  if and only if  $x - \hat{x}$  is orthogonal to every eigenvector b of the adjoint problem

$$\dot{x}(t) = -\frac{\partial f}{\partial x}(\hat{x})' x(t),$$

where b corresponds to an eigenvalue  $\xi$  of the Jacobian matrix at  $\hat{x}$  with  $\xi > 0$ . Then

$$\mathbf{E}^{s}(\hat{x}) = \{x : F'(x - \hat{x}) = 0\}, \qquad (C.6)$$

with F an  $n_+ \times n$  matrix constituted of all eigenvectors  $b_j$  corresponding to  $\xi_j > 0, \ j = n_- + 1, \dots, n.$ 

For equilibria of the canonical system of problems (DIP) the following theorem holds (see, e.g., Grass et al., 2008): **Theorem 3** (Dimension of the stable manifold). Let  $\hat{x} \in \mathbb{R}^{2n}$  be an equilibrium of the canonical system of a problem (DIP) and  $J(\hat{x})$  the Jacobian matrix. Then the corresponding eigenvalues  $\xi_i, i = 1, ..., 2n$ , are symmetric around r/2 and the dimension of the (local) stable manifold is at most n and

$$\operatorname{tr} J(\hat{x}) = rn$$

#### Appendix D. Numerical Continuation

We consider nonlinear (operator) equations of the form

$$F(x,\gamma) = 0 \tag{D.1}$$

where  $F: X \times \mathbb{R} \to Y$  is sufficiently smooth and X and Y are Banach-spaces, and a pair  $(x, \mu)$  satisfying (D.1) is called a *solution of* (D.1).<sup>8</sup> An important example for such a nonlinear operator equation is the BVP (C.1)-(C.2).

The task of *continuation* (*path-following*) is now, given a specific solution  $(x_s, \mu_s)$ , find a (smooth) solution curve  $x(\gamma)$  satisfying

$$F(x(\gamma), \gamma) = 0$$

with  $\mu \in [\mu_s, \mu_e]$ .

The existence of such a solution curve  $x(\gamma)$  is, e.g., assured by the implicit function theorem, existence theorems under less restrictive conditions can be found in Dontchev and Rockafellar (2009). For continuation in context with ODEs and BVPs the reader is referred to Kuznetsov (1998); Winkler (1985); Kitzhofer et al. (2009) and Krauskopf et al. (2007).

The numerical task of a *continuation process* is to provide an algorithm allowing the successive computation of points  $x(\mu_i)$ ,  $i = 1, \ldots, N$ , approximating the solution curve  $x(\mu)$  with  $\mu_N = \mu_e$ .

# Appendix D.1. Continuation Algorithms

Next we present two simple continuation algorithms working "above" some zero finding solver, in our context BVP solver (for a more detailed description see, e.g., Grass et al., 2008). This approach has the advantage of being independent on the actually used solver, but the disadvantage of disregarding structural information of the specific solver. Anyhow, even for the "above" method a more sophisticated continuation algorithm can be implemented, cf. Winkler (1985), but the following proved sufficient for all problems we analyzed so far.

The problem we are facing is

$$F(x,\mu) = 0, \tag{D.2}$$

with  $F: C^1 \times \mathbb{R}^p \to C^0$  a BVP, depending on the parameter value(s)  $\mu$ . Then a given solution  $(x_s, \mu_s)$  shall be continued to some  $\mu_e \neq \mu_s$ .

<sup>&</sup>lt;sup>8</sup>A reader unfamiliar with functional analysis can replace Banach-spaces by Euclidean spaces. In fact since for numerical purposes infinite dimensional spaces are discretized, the actual computation takes place in Euclidean spaces.

**Reparameterization:** We reparameterize the problem by introducing a scalar continuation variable  $\gamma \in [0, 1]$  setting

$$\mu(\gamma) := \mu_s + \gamma(\mu_e - \mu_s)$$
  
=  $\mu_s(1 - \gamma) + \gamma \mu_e, \ \gamma \in [0, 1],$  (D.3)

with  $\mu(0) = \mu_s$  and  $\mu(1) = \mu_e$ .

**Initialization:** We define some positive constant  $0 < \sigma_0 < 1$  and determine the solution of (D.1) for

$$\gamma_1 = \gamma_0 + \sigma_0, \tag{D.4}$$

with  $x_0 = x_s$  and  $\gamma_0 = 0$ .

**Continuation Step** i > 1: To follow solutions, where the stable manifold exhibits turning points we augment problem (D.2) by adding

$$\Phi(\gamma, \gamma_i, \sigma) = \gamma - \gamma_i - \sigma \tag{D.5}$$

$$\Phi(x, \gamma, x_i, \gamma_i, \sigma) := \|x - x_i\|^2 + (\gamma - \gamma_i)^2 - \sigma^2,$$
(D.6)

$$\Phi(x,\gamma,x_i,x_{i-1},\gamma_i,\gamma_{i-1},\sigma) := \begin{pmatrix} \Delta x_i \\ \Delta \gamma_i \end{pmatrix}' \begin{pmatrix} x-x_i \\ \gamma-\gamma_i \end{pmatrix} - \sigma \left\| \begin{pmatrix} \Delta x_i \\ \Delta \gamma_i \end{pmatrix} \right\|^2, \quad (D.7)$$
$$\Delta x_i = x_i - x_{i-1}, \ \Delta \gamma_i = \gamma_i - \gamma_{i-1},$$

where  $x_k = x(\gamma_k)$ , k = i - 1, *i* are previously detected solutions and  $\sigma > 0$ is a given constant, assuring that the new solution differs from the previous solution. Geometrically (D.6) describes a circle of radius  $\sigma$  around the solution at step *i*, whereas (D.7) ascribes a line perpendicular to the direction of the linear extrapolation of the last two detected solutions.

**Prediction Step:** The approximated solution is linearly extrapolated from the two previous solutions,  $(x_{i-1}, \gamma_{i-1})$  and  $(x_i, \gamma_i)$ , yielding

$$\tilde{x}_{i+1} = x_i + \alpha \left( x_i - x_{i-1} \right) = x_i (\alpha + 1) - \alpha x_{i-1}$$
 (D.8a)

$$\tilde{\gamma}_{i+1} = \gamma_i + (\gamma_i - \gamma_{i-1}) = \gamma_i(\alpha + 1) - \alpha \gamma_{i-1}.$$
 (D.8b)

The constant  $\alpha$  is determined, depending on the current and previous step size given by the ratio

$$\alpha = \frac{\sigma_{i+1}}{\sigma_i}.$$

# Appendix D.2. MATLAB package OCMat

It was also one of the aims of the author to implement the here presented method within a consistent framework allowing extension in various directions and making it accessible for anyone who is interested. Therefore a first step has been done by programming, together with A. Seidl,  $^9$  the MATLAB package  $\tt OCMat.^{10}$ 

The main steps for using this package are:

- 1. Provide an initialization file with the model description.
- 2. Generate the necessary conditions and needed files for computation automatically.
- 3. Create an MATLAB object consisting of the model for specific parameter values.
- 4. Analysis of the model, including
  - Calculation of equilibria, limit cycles.
  - Calculation of paths of the canonical system.
  - Locating and continuing indifference points.
  - Bifurcation analysis of the optimal vector field.
  - Interface to the numerical bifurcation tool MatCont, see Dhooge et al. (2003).<sup>11</sup>
- 5. Support to save and present the results
  - Plotting commands depicting solution paths in the phase space, as time paths, etc.
  - Storing the computational results for later usage.
  - Supporting the generation of animations.
  - Interface for creating figures under LaTeX.

In the actual version of OCMat models of the following type can be analyzed; autonomous models over a finite or infinite time horizon, mixed control constraints, pure state constraints of order one, and integral constraints all of inequality-type, i.e.,

$$c(x, u) \ge 0$$
,  $h(x) \ge 0$ ,  $\int_0^t i(x(s), u(s)) dt \ge 0$ .

The dynamics have to given by ODEs, the control variables have to appear nonlinearly, the finite number of optimal control values have to identified either explicitly or implicitly. Event though there is no principal restriction on the number of states and controls, this numerical technique is intended for rather low dimensional problems.

In updated versions of this package we want to increase the user-friendliness and also the range of possible applications. One of the strong points of the implementation as a MATLAB package is the capability of visualization as can be seen by the animations presented in this article.

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<sup>&</sup>lt;sup>10</sup>This can be downloaded from http://orcos.tuwien.ac.at/research/ocmat\_software.

<sup>&</sup>lt;sup>11</sup>This toolbox can be downloaded from http://matcont.sourceforge.net.

#### References

- Caulkins, J., Feichtinger, G., Grass, D., Hartl, R., Kort, P., 2010. Two state capital accumulation with heterogeneous products: disruptive vs. non-disruptive goods. Journal ofEconomicDynamics and Control In submission.
- Caulkins, J., Feichtinger, G., Grass, D., Johnson, M., Tragler, G., Yegorov, Y., 2005a. Placing the poor while keeping the rich in their place: Separating strategies for optimally managing residential mobility and assimilation. Demographic Research 13, 1–34.
- Caulkins, J., Feichtinger, G., Grass, D., Tragler, G., 2005b. A model of moderation: Finding Skiba points on a slippery slope. Central European Journal of Operations Research 45–64, 13.
- Caulkins, J., Feichtinger, G., Grass, D., Tragler, G., 2007. Bifurcating DNS thresholds in a model of organizational bridge building. Journal of Optimization Theory and Applications 133, 19–35.
- Caulkins, J., Feichtinger, G., Grass, D., Tragler, G., 2009. Optimal control of terrorism and global reputation: A case study with novel threshold behavior. Operations Research Letters 37, 387–391.
- Caulkins, J., Grass, D., Feichtinger, G., Tragler, G., 2008. Optimizing counterterror operations: Should one fight fire with "fire" or "water"? Computers & Operations Research 35, 1874–1885.
- Clark, C., 1976. Mathematical Bioeconomics. The Optimal Management of Renewable Resources. John Wiley & Sons, New York.
- Crépin, A.S., 2007. Using fast and slow processes to manage resources with thresholds. Environmental and Resource Economics 36, 191–213.
- Dhooge, A., Govaerts, W., Kuznetsov, Y., 2003. MATCONT: A Matlab package for numerical bifurcation analysis of ODEs. ACM Transactions on Mathematical Software (TOMS) 29, 141–164.
- Dontchev, A., Rockafellar, R., 2009. Implicit Functions and Solution Mappings: A View from Variational Analysis. Springer Verlag.
- Grass, D., Caulkins, J., Feichtinger, G., Tragler, G., Behrens, D., 2008. Optimal Control of Nonlinear Processes: With Applications in Drugs, Corruption, and Terror. Springer Verlag, Berlin.
- Kiseleva, T., Wagener, F., 2010. Bifurcations of optimal vector fields in the shallow lake system. Journal of Economic Dynamics and Control 34, 825– 843.
- Kitzhofer, G., Koch, O., Weinmüller, E., 2009. Pathfollowing for essentially singular boundary value problems with application to the complex Ginzburg-Landau equation. BIT Numerical Mathematics 49, 141–160.

- Krauskopf, B., Osinga, H., Galán-Vioque, J. (Eds.), 2007. Numerical Continuation Methods for Dynamical Systems: Path following and boundary value problems. Springer Verlag, Dordrecht.
- Kuznetsov, Y., 1998. Elements of Applied Bifurcation Theory. Springer Verlag, New York. 2nd edition.
- Levy, A., Neri, F., Grass, D., 2006. Macroeconomic aspects of substance abuse: Diffusion, productivity and optimal control. Macroeconomic Dynamics 10, 145–16.
- Michel, P., 1982. On the transversality condition in infinite horizon optimal problems. Econometrica 50, 975–985.
- Wagener, F., 2003. Skiba points and heteroclinic bifurcations, with applications to the shallow lake system. Journal of Economic Dynamics and Control 27, 1533–1561.
- Wagener, F., 2006. Skiba points for small discount rates. Journal of Optimization Theory and Applications 128, 261–277.
- Winkler, R., 1985. Path-following for two-point boundary value problems. Seminarbericht 78. Department of Mathematics, Humboldt-University Berlin, Germany.
- Zeiler, I., Caulkins, J., Grass, D., Tragler, G., 2010. Keeping options open: an optimal control model with trajectories that reach a DNSS point in positive time. SIAM Journal on Control and Optimization 48, 3698–3707.